Math 6880 : Fluid Dynamics I

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Contents

Preface

These notes are largely based on **Math 6880: Fluid Dynamics** course, taught by Aaron Fogelson and Christel Hohenegger in Fall 2017 and Spring 2018, at the University of Utah. Additional examples or remarks or results from other sources are added as I see fit, mainly to facilitate my understanding. These notes are by no means accurate or applicable, and any mistakes here are of course my own. Please report any typographical errors or mathematical fallacy to me by email tan@math.utah.edu.

To-do list:

2.4 Figure of tetrahedron with stress tensor components.

2.7.3 Figure of deformed cube.

3.2.2 Interpret the solution.

1. CHT:... (water waves), which are easily seen by everyone and which are usually used as an example of waves in elementary courses... are the worst possible example... they have all the complications that waves can have. —– Richard Feynman, The Feynman Lectures on Physics.

Chapter 1

Tensor Algebra and Calculus

The physical quantities encountered in fluid mechanics can be divided into three categories:

1. **scalars** (zero-order tensors) such as shear rate, energy, volume and time;
2. **vectors** (first-order tensors) such as velocity, momentum and force;
3. **second-order tensors** such as stress and rate of strain tensors.

In this chapter we briefly review vector calculus and then extend these to tensor calculus on tensor fields. Various important concepts include gradient, divergence, curl, Laplacian and the divergence theorem. We refer the interested reader to [GS08] for an excellent introduction.

1.1 Cartesian Tensors

A Cartesian tensor uses an orthonormal basis to represent a vector in a Euclidean space in the form of components. Specifically, let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ denote the standard basis vectors in $\mathbb{R}^3$. Any vector $\boldsymbol{v} \in \mathbb{R}^3$ has a unique decomposition

$$\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3 \tag{1.1.1}$$

and the scalars v_1, v_2, v_3 are called the components (coordinates) of $\boldsymbol{v}$ in the standard basis.

1.1.1 Summation convention

Because we always use Cartesian tensors and operations on these tensors in terms of components naturally involves sums, we adopt a convention, called the **Einstein notation**:

Whenever an index appears twice in a term,
a sum is implied over that index.

In other words, we are summing over repeated indices, and the number of repetition depends on the dimension of the tensor. For example, (1.1.1) in Einstein notation is simply

$$\boldsymbol{v} = v_i\boldsymbol{e}_i.$$

The summation convention applies only *pairs of repeated indices* within any expressions. In particular, terms of the form a_i, $a_ib_ic_i$ and so on are meaningless. There are two possible types of indices:

1. one that appears twice, called a **dummy index**;

2. one that appears once, called a **free index**.

For example, in the equation

$$a_i = b_i + \sum_{j=1}^{3} c_{ij},$$

the index i is the free index while j is the dummy index. Note that each term in an equation must be consistent in terms of the free index, *i.e.* each term should have the same free indices.

1.1.2 Kronecker delta and permutation symbols

The *Kronecker delta* is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

For $i, j, k \in \{1, 2, 3\}$, it is easy to verify that

$$\delta_{ii} = 3, \quad \delta_{ij}\delta_{ij} = 3, \quad \delta_{ij}\delta_{jk} = \delta_{ik}.$$

We can express the dot product in terms of δ_{ij}:

$$\boldsymbol{u} \cdot \boldsymbol{v} = (u_i \boldsymbol{e}_i) \cdot (v_j \boldsymbol{e}_j) = u_i v_j \delta_{ij} = u_i v_i.$$

Also, if $\boldsymbol{x}$ is a vector with coordinates x_1, x_2, x_3, then the partial derivative sof x with respect to these coordinates can be defined as

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}.$$

The *permutation symbol* (or more generally the Levi-Civita symbol) ε_{ijk} is defined as

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123,\ 231 \text{ or } 312, \\ -1 & \text{if } ijk = 321,\ 213 \text{ or } 132, \\ 0 & \text{otherwise (repeated index).} \end{cases}$$

It is evident that $\varepsilon_{ijk}\varepsilon_{ijk} = 6$. We can express the vector cross product in terms of ε_{ijk}:

$$\boldsymbol{u} \times \boldsymbol{v} = (u_i \boldsymbol{e}_i) \times (v_j \boldsymbol{e}_j) = u_i v_j (\boldsymbol{e}_i \times \boldsymbol{e}_j) = u_i v_j \varepsilon_{ijk} \boldsymbol{e}_k. \tag{1.1.2}$$

The determinant of a 3×3 matrix A are also related to the permutation symbol:

$$\det(A) = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k} = |\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)|,$$

where $\boldsymbol{a}_i$ is the ith column of the matrix A.

A *permutation* is the act of rearranging members of a given set into some other order. A *tranposition* is a permutation in which two adjacent indices are interchanged. An *even* (*odd*) permutation is a permutation that can be achieved in an even (odd) number of tranpositions. With these definitions, observe the following:

1. δ_{ij} is invariant under transposition of indices, whereas ε_{ijk} change sign under (pairwise) transposition, *e.g.* $\varepsilon_{ijk} = -\varepsilon_{jik}$;

2. ε_{ijk} is invariant under circular or cyclic permutation of indices, in the sense that $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$.

The Kronecker delta and the permutation symbol satisfy the following simple identity, which can be used to prove many vector identities with ease!

Proposition 1.1.1 (Epsilon-Delta Identity)**.** *Let δ_{ij} be the Kronecker delta and ε_{ijk} the permutation symbol. Then*

$$\varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{jp}\delta_{iq} \quad \textit{and} \quad \varepsilon_{mjk}\varepsilon_{njk} = 2\delta_{mn}.$$

As an application of the Epsilon-Delta identity, we prove the following identity involving the triple vector product

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}.$$

Using (1.1.2), we have

$$\begin{aligned}
\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= (a_m\boldsymbol{e}_m) \times (b_i c_j \varepsilon_{ijk}\boldsymbol{e}_k) \\
&= a_m b_i c_j \varepsilon_{ijk}\varepsilon_{mkn}\boldsymbol{e}_n \\
&= -a_m b_i c_j \varepsilon_{ijk}\varepsilon_{mnk}\boldsymbol{e}_n \\
&= -a_m b_i c_j \Big[\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\Big]\boldsymbol{e}_n \\
&= -a_i b_i c_n \boldsymbol{e}_n + a_j c_j b_n \boldsymbol{e}_n \\
&= (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}.
\end{aligned}$$

1.2 Second-Order Tensor

A second-order tensor $\underline{\underline{T}}$ represents a linear function which takes a vector as input and gives a vector as output. Since we are mostly dealing with vectors in $\mathbb{R}^3$,

Definition 1.2.1. A second-order tensor $\underline{\underline{T}}$ on $\mathbb{R}^3$ is a linear transformation $\underline{\underline{T}}\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that

$$\underline{\underline{T}}(\alpha\boldsymbol{u} + \beta\boldsymbol{v}) = \alpha\underline{\underline{T}}\boldsymbol{u} + \beta\underline{\underline{T}}\boldsymbol{v} \quad \text{for any } \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3 \text{ and scalars } \alpha, \beta \in \mathbb{R}.$$

In describing the behaviour of material bodies we will also require the concept of a *linear transformation between second-order tensors*. This leads to the notion of a *fourth-order tensor* but we defer this discussion until later chapters and concentrate on second-order tensors. Analogous to representing vectors in terms of standard basis vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ in $\mathbb{R}^3$, we can represent any second-order tensor as a linear combination of the nine second-order tensors obtained by forming outer products $\boldsymbol{e}_i\boldsymbol{e}_j^T$. In *dyadic form*,

$$\boldsymbol{e}_i\boldsymbol{e}_j^T = \boldsymbol{e}_i\boldsymbol{e}_j = \boldsymbol{e}_i \otimes \boldsymbol{e}_j.$$

For example,

$$\boldsymbol{e}_1\boldsymbol{e}_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}.$$

We then have

$$\underline{\underline{T}} = T_{ij}\boldsymbol{e}_i\boldsymbol{e}_j = \begin{bmatrix}T_{11} & T_{12} & T_{13}\\T_{21} & T_{22} & T_{23}\\T_{31} & T_{32} & T_{32}\end{bmatrix}$$

and the scalars T_{ij} are the components of $\underline{\underline{T}}$ with respect to the standard basis dyads $\boldsymbol{e}_i\boldsymbol{e}_j$ in $\mathbb{R}^{3\times3}$.

1.2.1 Tensor algebra

Recall the following dyadic algebra:

$$(\boldsymbol{e}_i\boldsymbol{e}_j)\cdot\boldsymbol{e}_k = \boldsymbol{e}_i(\boldsymbol{e}_j\cdot\boldsymbol{e}_k) \quad \text{and} \quad (\boldsymbol{e}_i\boldsymbol{e}_j)\times\boldsymbol{e}_k = \boldsymbol{e}_i(\boldsymbol{e}_j\times\boldsymbol{e}_k).$$

There are three basic operations on first- and second-order tensors:

1. The inner product $\cdot$ which retains or reduces the rank of the tensors;

$$\begin{aligned}
\underline{\underline{T}}\cdot\boldsymbol{u} &= (T_{ij}\boldsymbol{e}_i\boldsymbol{e}_j)\cdot(u_k\boldsymbol{e}_k) &&= T_{ij}u_k\delta_{jk}\boldsymbol{e}_i &&= T_{ij}u_j\boldsymbol{e}_i && \text{(Rank 1)}\\
\boldsymbol{u}\cdot\underline{\underline{T}} &= (u_i\boldsymbol{e}_i)\cdot(T_{jk}\boldsymbol{e}_j\boldsymbol{e}_k) &&= T_{jk}u_i\delta_{ij}\boldsymbol{e}_k &&= T_{jk}u_j\boldsymbol{e}_k && \text{(Rank 1)}\\
\underline{\underline{T}}\cdot\underline{\underline{S}} &= (T_{ij}\boldsymbol{e}_i\boldsymbol{e}_j)\cdot(S_{kl}\boldsymbol{e}_k\boldsymbol{e}_l) &&= T_{ij}S_{kl}\delta_{jk}\boldsymbol{e}_i\boldsymbol{e}_k &&= T_{ij}S_{jl}\boldsymbol{e}_i\boldsymbol{e}_l. && \text{(Rank 2)}
\end{aligned}$$

   These correspond to standard matrix-vector and matrix-matrix multiplication.

2. The double inner product or *contraction* $:$ which reduces the rank of the tensors;

$$\begin{aligned}
\underline{\underline{T}} : \underline{\underline{S}} &= T_{ij}S_{ij} && \text{(Rank 0)}\\
\varepsilon : T &= \varepsilon_{ijk}T_{jk}\boldsymbol{e}_i = \varepsilon_{jki}T_{jk}\boldsymbol{e}_i = \varepsilon_{kij}T_{jk}\boldsymbol{e}_i. && \text{(Rank 1)}
\end{aligned}$$

3. The outer product or *tensor product* $\otimes$ which retains or increases the rank of the tensor;

$$\boldsymbol{u}\otimes\underline{\underline{T}} = (u_i\boldsymbol{e}_i)\otimes(T_{jk}\boldsymbol{e}_j\boldsymbol{e}_k) = u_iT_{jk}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k. \qquad \text{(Rank 3)}$$

A tensor $\underline{\underline{S}} \in \mathbb{R}^{3\times3}$ is *symmetric* if $\underline{\underline{S}}^T = \underline{\underline{S}}$ and *skew-symmetric* if $\underline{\underline{S}}^T = -\underline{\underline{S}}$. It follows that every second-order tensor $\underline{\underline{S}} \in \mathbb{R}^{3\times3}$ can be uniquely written as

$$\underline{\underline{S}} = \underline{\underline{E}} + \underline{\underline{W}},$$

where $\underline{\underline{E}}$ and $\underline{\underline{W}}$ are symmetric and skew-symmetric tensors respectively, having the form

$$\underline{\underline{E}} = \frac{1}{2}\left(\underline{\underline{S}} + \underline{\underline{S}}^T\right) \quad \text{and} \quad \underline{\underline{W}} = \frac{1}{2}\left(\underline{\underline{S}} - \underline{\underline{S}}^T\right).$$

1.2.2 Isotropic tensor

A tensor is said to be **isotropic** if its components do not change when we rotate the coordinate system. For a second-order tensor $\underline{\underline{T}}$, it is said to be isotropic if

$$\underline{\underline{Q}}\,\underline{\underline{T}}\,\underline{\underline{Q}}^T = \underline{\underline{T}} \quad \text{for all rotations } \underline{\underline{Q}}.$$

All rank 0 tensors are isotropic but there are no nonzero rank 1 isotropic tensors.

Proposition 1.2.2. *The only rank 2 isotropic tensor is $T_{ij} = \alpha\delta_{ij}$ for any scalars $\alpha \in \mathbb{R}$. The only rank 4 isotropic tensor is*

$$a\delta_{ij}\delta_{pq} + b\delta_{ip}\delta_{jq} + c\delta_{iq}\delta_{jp} \tag{1.2.1}$$

for any scalars $a, b, c \in \mathbb{R}$. An alternate version of (1.2.1) is more useful as we shall see later:

$$\alpha\delta_{ij}\delta_{pq} + \beta\left(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}\right) + \gamma\left(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}\right), \tag{1.2.2}$$

where α, β, γ are all scalars.

Proof. Consider a general rank 2 tensor $\underline{\underline{T}}$ with components T_{ij} with respect to some coordinate frame $\{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$ and suppose that it is isotropic. Consider the 90° counter-clockwise rotation about the $\boldsymbol{f}_3$-axis and $\boldsymbol{f}_2$-axis, which can be expressed in terms of the second-order tensor $\underline{\underline{S}}_3$ and $\underline{\underline{S}}_2$ respectively:

$$\underline{\underline{S}}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \underline{\underline{S}}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Since $\underline{\underline{T}}$ is isotropic, the following must be true:

$$\underline{\underline{S}}_3\,\underline{\underline{T}}\,\underline{\underline{S}}_3^T = \underline{\underline{T}} = \underline{\underline{S}}_2\,\underline{\underline{T}}\,\underline{\underline{S}}_2^T.$$

Expanding these yields

$$\begin{bmatrix} T_{22} & -T_{21} & -T_{23} \\ -T_{12} & T_{11} & T_{13} \\ -T_{32} & T_{31} & T_{33} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} T_{33} & T_{32} & -T_{31} \\ T_{23} & T_{22} & -T_{21} \\ -T_{13} & -T_{12} & T_{11} \end{bmatrix}.$$

Comparing the first two matrices, we see that $T_{11} = T_{22}$ and

$$\begin{aligned} -T_{23} = T_{13} = T_{23} &\implies T_{23} = T_{13} = 0 \\ -T_{32} = T_{31} = T_{32} &\implies T_{32} = T_{31} = 0. \end{aligned}$$

Comparing the last two matrices, we see that $T_{11} = T_{33}$ and $T_{12} = T_{32} = 0$, $T_{21} = T_{23} = 0$. Therefore all the off-diagonal elements of $\underline{\underline{T}}$ are zero and all the diagonal elements are equal, say α. The claim follows. ■

1.2.3 Gradient, divergence, curl and Laplacian

Suppose $\underline{\underline{T}} = \underline{\underline{T}}(\boldsymbol{x})$ is a spatial-dependent tensor field. Recall the del operator ∇ in Cartesian coordinates

$$\nabla = \frac{\partial}{\partial x_i}\boldsymbol{e}_i = \partial_{x_i}\boldsymbol{e}_i = \partial_i \boldsymbol{e}_i.$$

The **gradient** of a tensor field $\underline{\underline{T}}$ of any rank is defined as

$$\nabla \underline{\underline{T}} = \nabla \otimes \underline{\underline{T}}.$$

Note that $\nabla \underline{\underline{T}}$ is one rank higher than $\underline{\underline{T}}$. For a scalar function $\phi(\boldsymbol{x})$,

$$\nabla \phi = (\partial_i \boldsymbol{e}_i) \otimes \phi = \partial_i \phi \boldsymbol{e}_i.$$

For a vector field $\boldsymbol{u}(\boldsymbol{x})$,

$$\nabla \boldsymbol{u} = (\partial_i \boldsymbol{e}_i) \otimes (u_j \boldsymbol{e}_j) = \partial_i u_j \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \partial_i u_j \boldsymbol{e}_i \boldsymbol{e}_j.$$

The **divergence** of a tensor field $\underline{\underline{T}}$ of any rank $n \geq 2$ is defined as

$$\text{div } \underline{\underline{T}} = \nabla \cdot \underline{\underline{T}}.$$

Note that div $\underline{\underline{T}}$ is one rank lower than $\underline{\underline{T}}$. For a vector field $\boldsymbol{u}(\boldsymbol{x})$,

$$\nabla \cdot \boldsymbol{u} = (\partial_i \boldsymbol{e}_i) \cdot (u_j \boldsymbol{e}_j) = \partial_i u_j (\boldsymbol{e}_i \cdot \boldsymbol{e}_j) = \partial_i u_j \delta_{ij} = \partial_i u_i.$$

For a rank 2 tensor field $\underline{\underline{T}}(\boldsymbol{x})$,

$$\nabla \cdot \underline{\underline{T}} = (\partial_i \boldsymbol{e}_i) \cdot (T_{jk} \boldsymbol{e}_j \boldsymbol{e}_k) = \partial_i T_{jk} \delta_{ij} \boldsymbol{e}_k = \partial_i T_{ik} \boldsymbol{e}_k,$$

i.e. $\nabla \cdot \underline{\underline{T}} = (\nabla \cdot T_1, \nabla \cdot T_2, \nabla \cdot T_3)^T$ with T_i the ith column of $\underline{\underline{T}}$.

The **curl** of a vector field $\boldsymbol{u}(\boldsymbol{x})$ is given by

$$\begin{aligned} \nabla \times \boldsymbol{u} = (\partial_m \boldsymbol{e}_m) \times (u_n \boldsymbol{e}_n) &= \partial_m u_n \varepsilon_{ijk} e_{m,i} e_{n,j} \boldsymbol{e}_k \\ &= \partial_m u_n \varepsilon_{ijk} \delta_{im} \delta_{jn} \boldsymbol{e}_k \\ &= \partial_m u_n \varepsilon_{mnk} \boldsymbol{e}_k. \end{aligned}$$

The vector field $\boldsymbol{u}$ is said to be **irrotational** if $\nabla \times \boldsymbol{u} \equiv \boldsymbol{0}$. Interpreting $\boldsymbol{u}$ as the fluid velocity field, the curl at a point $\boldsymbol{x}$ provides information on the direction and angular speed of the rotation at $\boldsymbol{x}$. Last but not least, we introduce the **Laplacian** operator which is simply the composition of the divergence and gradient operator, *i.e.*

$$\Delta = \nabla^2 = \nabla \cdot \nabla.$$

For a scalar function $\phi(\boldsymbol{x})$, the scalar Laplacian is

$$\Delta \phi = \nabla \cdot \nabla \phi = (\partial_i \boldsymbol{e}_i) \cdot (\partial_j \phi \boldsymbol{e}_j) = \partial_i \partial_j \phi \delta_{ij} = \partial_i \partial_i \phi = \partial_i^2 \phi.$$

For a vector field $\boldsymbol{u}(\boldsymbol{x})$, the vector Laplacian is

$$\Delta \boldsymbol{u} = \nabla \cdot \nabla \boldsymbol{u} = (\partial_i \boldsymbol{e}_i) \cdot (\partial_j u_k \boldsymbol{e}_j \boldsymbol{e}_k) = \partial_i \partial_j u_k \delta_{ij} \boldsymbol{e}_j = \partial_i \partial_i u_k \boldsymbol{e}_k,$$

i.e. $\Delta \boldsymbol{u} = (\Delta u_1, \Delta u_2, \Delta u_3)^T$. For completeness, we state without proof the following proposition involving several vector identities.

Proposition 1.2.3. *Let ϕ be a scalar function and $\boldsymbol{u}$ a vector field. Then*

$$\begin{aligned}\nabla \cdot (\phi \boldsymbol{u}) &= \nabla\phi \cdot \boldsymbol{u} + \phi \nabla \cdot \boldsymbol{u} \\ \nabla \times (\nabla \phi) &= 0 \\ \nabla \cdot (\nabla \times \boldsymbol{u}) &= 0 \\ \nabla \times (\nabla \times \boldsymbol{u}) &= \nabla(\nabla \cdot \boldsymbol{u}) - \Delta \boldsymbol{u}.\end{aligned}$$

1.3 Generalised Divergence Theorem

We state without proof the divergence theorem for a vector field, which is one of the most important result in vector calculus.

Theorem 1.3.1 (Divergence Theorem)**.** *Let V be a Lipschitz domain in $\mathbb{R}^3$ with piecewise smooth boundary $\partial V = S$. If $\boldsymbol{u}$ is a C^1 vector field defined on a neighbourhood of V, then*

$$\int_V \nabla \cdot \boldsymbol{u}\, dV = \int_S \boldsymbol{u} \cdot \boldsymbol{n}\, dS,$$

where $\boldsymbol{n}$ is the outward unit normal vector on S. The integral on the right is the flux *of $\boldsymbol{u}$ across the boundary (oriented surface) S.*

One can generalised the divergence theorem to second-order tensor fields.

Theorem 1.3.2 (Generalised Divergence Theorem)**.** *Let V be a Lipschitz domain in $\mathbb{R}^3$ with piecewise smooth boundary $\partial V = S$. If $\underline{\underline{T}}$ is a second-order tensor field defined on a neighbourhood of V, then*

$$\int_V \nabla \cdot \underline{\underline{T}}\, dV = \int_S \boldsymbol{n} \cdot \underline{\underline{T}}\, dS,$$

where $\boldsymbol{n}$ is the outward unit normal vector on S.

Proof. Let $\boldsymbol{n} = (n_1, n_2, n_3)^T$. The main idea is to apply the divergence theorem to each component of $\nabla \cdot \underline{\underline{T}}$:

$$\int_V \nabla \cdot \underline{\underline{T}}\, dV = \int_V \partial_j T_{jk} \boldsymbol{e}_k\, dV = \int_A n_j T_{jk} \boldsymbol{e}_k\, dA = \int_A \boldsymbol{n} \cdot \underline{\underline{T}}\, dA.$$

For the last equality, note that

$$\boldsymbol{n} \cdot \underline{\underline{T}} = (n_j \boldsymbol{e}_j) \cdot (T_{ik} \boldsymbol{e}_i \boldsymbol{e}_k) = n_j T_{ik} \delta_{ji} \boldsymbol{e}_k = n_j T_{jk} \boldsymbol{e}_k.$$

■

Chapter 2

Navier-Stokes Equations

Fluids such as gases and liquids do not have a fixed form or shape in contrast with solid, *i.e.* there is no preferred rest state. Nonetheless, fluid offers resistance to an imposed force when due this force its form changes or when the fluid starts to flow. The so-called *simple fluids* are those which have the property that forces are linearly proportional to the rate of deformation; these fluids are said to be Newtonian. There are also fluids in which the forces depend nonlinearly on the rate of deformation and viscoelastic fluids that combine the properties of an elastic solid and a simple fluid, where they react as a solid to fast deformations and as a fluid to slow deformations, depending on the characteristic time scale of the material. These are called *complex fluids*.

At a *microscopic* scale, fluids are made up of individual molecules. As an example, there are about $3.3.46 \times 10^{25}$ water molecules in a litre of water. Although its physical properties are violently nonuniform, in most situations we are concerned with a *macroscopic* description of a fluid motion and this leads to the **continuum hypothesis**, which states that the fluid material can be treated as perfectly continuous. Consequently, at each point of the fluid we can define, by averaging over a small volume, macroscopic properties such as density, pressure and bulk velocity and that these vary smoothly over the fluid. We shall use the term *fluid particle* or *fluid element* to indicate such a small volume, and small in this case means that the characteristic length scale related to the volume is small compared to the length scale of the fluid motion, but large compared to the characteristic molecular scale such as the mean free path in a gas or the intermolecular distance between molecules in a liquid.

The main goal of this chapter is to derive equations of motion for the fluid velocity on the continuum level, and these will necessarily have the form of *partial differential equations (PDEs)*.

2.1 Flow Maps and Kinematics

The description of the fluid motion is called kinematics, and we are interested in the kinematics of a continuum of fluid. Specifically, we want to mathematically describe the displacement, velocity and acceleration of fluid material points in the two reference frames commonly used in fluid mechanics, as we discuss in detail in a moment.

2.1.1 Lagrangian and Eulerian descriptions

Let $\mathcal{D} \subset \mathbb{R}^3$ be a region filled with fluid. Assume that at time $t = 0$, the location of a fluid particle is denoted by $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$. We can describe the location of such fluid particle $\boldsymbol{x} = (x_1, x_2, x_3)$ at any time $t \geq 0$ by a *flow map* $\boldsymbol{\phi}$, defined as

$$\begin{aligned} \boldsymbol{\phi} \colon \mathbb{R}^3 \times \mathbb{R} &\longrightarrow \mathbb{R}^3 \\ (\boldsymbol{\alpha}, t) &\mapsto \boldsymbol{x} \end{aligned}$$

and $\boldsymbol{\phi}(\boldsymbol{\alpha}, 0) = \boldsymbol{\alpha}$. We will assume that $\boldsymbol{\phi}$ is smooth and invertible, the latter means that given a time $t > 0$ and spatial location $\boldsymbol{x} \in \mathcal{D}$, we can identify the unique $\boldsymbol{\alpha}$ for the fluid, *i.e.*

$$\boldsymbol{\alpha} = \boldsymbol{\phi}^{-1}(\boldsymbol{x}, t).$$

The physical realisation is that no two fluid particles can occupy the same spatial location at the same time. Mathematically, invertibility means that the Jacobian $J = \det(D\boldsymbol{\phi})$ of the flow map is nonzero, *i.e.*

$$J(\boldsymbol{\alpha}, t) = \left| \frac{\partial(x_1, x_2, x_3)}{\partial(\alpha_1, \alpha_2, \alpha_3)} \right| = \begin{vmatrix} \partial_{\alpha_1} x_1 & \partial_{\alpha_2} x_1 & \partial_{\alpha_3} x_1 \\ \partial_{\alpha_1} x_2 & \partial_{\alpha_2} x_2 & \partial_{\alpha_3} x_2 \\ \partial_{\alpha_1} x_3 & \partial_{\alpha_2} x_3 & \partial_{\alpha_3} x_3 \end{vmatrix} \neq 0.$$

More generally, let $\Omega_0 \subset \mathcal{D}$ be a region of fluid at $t = 0$, which can be viewed as a "chunk" of fluid in $\mathcal{D}$. We can track the motion of Ω_0 using the flow map:

$$\begin{aligned} \Omega_t = \Omega(t) &= \boldsymbol{\phi}(\Omega_0, t) \\ &= \{\boldsymbol{x} \in \mathcal{D} \colon \boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{\alpha}, t) \ \text{ for some } \boldsymbol{\alpha} \in \Omega_0\}. \end{aligned}$$

Ω_t is called the material volume which is a volume moving with the fluid.

Definition 2.1.1. Given a flow map $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{\alpha}, t)$,

1. $\boldsymbol{\alpha}$ is called a **material** or **Lagrangian** coordinate; it describes a particular fluid particle;

2. $\boldsymbol{x}$ is called a **spatial** or **Eulerian** coordinate; it describes a particular location in space.

For the **Lagrangian description** we start with a fluid element and follow it through the fluid. The Lagrangian coordinate $\boldsymbol{\alpha}$ need not be the initial position of a fluid element, although that is the most common choice. Working in the Lagrangian frame has certain theoretical and mathematical advantages, but it is often difficult to apply in practice since any measurements in a fluid tend to be performed at fixed points in space as the fluid flows past the point. On the other hand, if we wish to observe fluid properties at a fixed location $\boldsymbol{x}$ as a function of time, we must realise that as time evolves different fluid elements will occupy the location $\boldsymbol{x}$. This constitutes the **Eulerian description** which is the most commonly used way of describing a fluid motion.

It is of interest to compare these two descriptions and explore their connections. Because the velocity at a location $\boldsymbol{x}$ and time t must be equal to the velocity of the fluid particle which is at this position and at this particular time, the Eulerian and Lagrangian coordinates are related as follows:

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}\left(\boldsymbol{\phi}\left(\boldsymbol{\alpha},t\right),t\right) = \frac{\partial \boldsymbol{\phi}}{\partial t}\left(\boldsymbol{\alpha},t\right), \tag{2.1.1}$$

where $\boldsymbol{u}$ is the fluid velocity field. Given the Eulerian velocity field, computing Lagrangian coordinates is therefore equivalent to solving (2.1.1) with initial condition $\boldsymbol{x}(0) = \boldsymbol{\phi}(\boldsymbol{\alpha},0) = \boldsymbol{\alpha}$.

Example 2.1.2. In one dimension, consider the velocity field given in Eulerian coordinates by $u(x,t) = \dfrac{2x}{1+t}$. The Lagrangian coordinate $\phi(\alpha,t)$ can be found by solving

$$\begin{cases} \dfrac{\partial \phi}{\partial t}(\alpha,t) = u(\phi(\alpha,t),t)) = \dfrac{2\phi}{1+t} \\ \phi(\alpha,0) = \alpha. \end{cases}$$

This is a separable ODE and its solution is

$$\phi(\alpha,t) = C(1+t)^2 = \alpha(1+t)^2.$$

The Lagrangian velocity as a function of α and t is

$$\frac{\partial \phi}{\partial t}(\alpha,t) = 2\alpha(1+t),$$

which can also be found by evaluating the Eulerian velocity at $x = \phi(\alpha,t)$.

2.1.2 Material derivative

Since we have different way in describing the flow, care must be taken in defining the "time derivative". Let $f(\boldsymbol{x},t)$ be some quantity of interest defined at each fluid particle, where $\boldsymbol{x}$ is understood to change with time at the local flow velocity $\boldsymbol{u}$, *i.e.*

$$\frac{dx_i}{dt} = u_i, \quad i = 1,2,3.$$

There are two notions of time derivative of f:

$$\begin{aligned} \frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial t}\right)_{\boldsymbol{x}\text{ fixed}} &= \text{time rate of change of } f \text{ at a fixed location } \boldsymbol{x}. \\ \frac{Df}{Dt} = \left(\frac{\partial f}{\partial t}\right)_{\boldsymbol{\alpha}\text{ fixed}} &= \text{time rate of change of } f \text{ for the fluid particle} \\ &\quad\ \text{which happens to be at location } \boldsymbol{x} \text{ at time } t \\ &= \textbf{material derivative} \text{ or } \textbf{Lagrangian derivative}. \end{aligned}$$

Using Chain Rule, we obtain an explicit relation between $\dfrac{\partial}{\partial t}$ and $\dfrac{D}{Dt}$:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial t}$$

$$= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}u_1 + \frac{\partial f}{\partial x_2}u_2 + \frac{\partial f}{\partial x_3}u_3$$
$$= \frac{\partial f}{\partial t} + \boldsymbol{u} \cdot \nabla f.$$

The difference between the usual time derivative and the material derivative is best illustrated with the following example. Assume that water is flowing through a pipe with a constriction and the motion is steady, *i.e.* the velocity at any spatial location $\boldsymbol{x}$ is not changing in time; this means that $\frac{\partial \boldsymbol{x}}{\partial t} = 0$. However, if we follow along a particular fluid particle, its material derivative is changing since its velocity is varying at different spatial location. Indeed, the velocity in the middle of the pipe is greater due to the geometry of the pipe, since the fluid flow is steady by assumption.

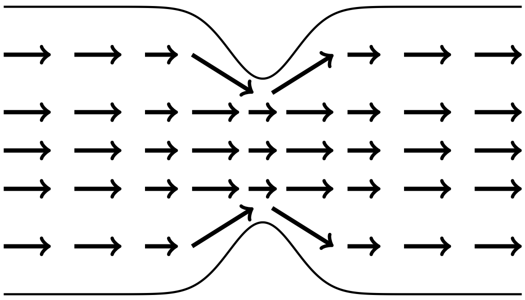

Figure 2.1: The direction field of a steady fluid through a constricted section of a pipe.

Example 2.1.3. Consider the flow field

$$\boldsymbol{u}(\boldsymbol{x}, t) = (xy)^2 \boldsymbol{e}_x + ze^{-\alpha t}\boldsymbol{e}_y + \cos(2xz)\boldsymbol{e}_z.$$

The fluid acceleration in an Eulerian frame is

$$\frac{\partial \boldsymbol{u}(\boldsymbol{x}, t)}{\partial t} = \frac{\partial}{\partial t}((xy)^2)\boldsymbol{e}_x + \frac{\partial}{\partial t}(ze^{-\alpha t})\boldsymbol{e}_y + \frac{\partial}{\partial t}(\cos(2xz))\boldsymbol{e}_z$$
$$= -\alpha z e^{-\alpha t}\boldsymbol{e}_y.$$

The fluid acceleration in a Lagrangian frame is given by $\frac{D\boldsymbol{u}(\boldsymbol{x}, t)}{Dt}$. Computing $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ gives

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = (u_i \boldsymbol{e}_i) \cdot (\partial_j u_k \boldsymbol{e}_j \boldsymbol{e}_k) = u_i \partial_j u_k \delta_{ij} \boldsymbol{e}_k$$
$$= u_j \partial_j u_k \boldsymbol{e}_k,$$

which results in

$$\frac{D\boldsymbol{u}(\boldsymbol{x}, t)}{Dt} = \frac{D(x^2y^2)}{Dt}\boldsymbol{e}_x + \frac{D(ze^{-\alpha t})}{Dt}\boldsymbol{e}_y + \frac{D(\cos(2xz))}{Dt}\boldsymbol{e}_z.$$

We are left with finding the material derivative of $\boldsymbol{u}(\boldsymbol{x},t)$ component-wise:

$$\begin{aligned}
\frac{D(x^2y^2)}{Dt} &= \frac{\partial(x^2y^2)}{\partial t} + \boldsymbol{u}\cdot\nabla(x^2y^2) \\
&= \boldsymbol{u}\cdot\left(2xy^2\boldsymbol{e}_x + 2x^2y\boldsymbol{e}_y\right) \\
&= 2x^3y^4 + 2x^2yze^{-\alpha t} \\
\frac{D(ze^{-\alpha t})}{Dt} &= \frac{\partial(ze^{-\alpha t})}{\partial t} + \boldsymbol{u}\cdot\nabla\left(ze^{-\alpha t}\right) \\
&= -\alpha ze^{-\alpha t} + \boldsymbol{u}\cdot\left(e^{-\alpha t}\boldsymbol{e}_z\right) \\
&= -\alpha ze^{-\alpha t} + e^{-\alpha t}\cos(2xz) \\
\frac{D(\cos(2xz))}{Dt} &= \frac{\partial(\cos(2xz))}{\partial t} + \boldsymbol{u}\cdot\nabla\left(\cos(2xz)\right) \\
&= \boldsymbol{u}\cdot\left(-2z\sin(2xz)\boldsymbol{e}_x - 2x\sin(2xz)\boldsymbol{e}_z\right) \\
&= -2x^2y^2z\sin(2xz) - 2x\sin(2xz)\cos(2xz) \\
&= -2x^2y^2z\sin(2xz) - x\cos(4xz).
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{D\boldsymbol{u}(\boldsymbol{x},t)}{Dt} &= \frac{D(x^2y^2)}{Dt}\boldsymbol{e}_x + \frac{D(ze^{-\alpha t})}{Dt}\boldsymbol{e}_y + \frac{D(\cos(2xz))}{Dt}\boldsymbol{e}_z \\
&= \left[2x^3y^4 + 2x^2yze^{-\alpha t}\right]\boldsymbol{e}_x \\
&\quad + \left[-\alpha ze^{-\alpha t} + e^{-\alpha t}\cos(2xz)\right]\boldsymbol{e}_y \\
&\quad + \left[-2x^2y^2z\sin(2xz) - x\cos(4xz)\right]\boldsymbol{e}_z.
\end{aligned}$$

Theorem 2.1.4 (Euler's Identity)**.** *Let J be the Jacobian of the flow map $\boldsymbol{\phi}$. The material derivative of J satisfies*

$$\frac{DJ}{Dt} = J\nabla\cdot\boldsymbol{u}. \tag{2.1.2}$$

Proof. Recall the **Jacobi's formula**: If $A(t)$ is an $n\times n$ matrix with real entries $(a_{ij}(t))$, then the derivative of its determinant $\det(A(t))$ is given by

$$\frac{d}{dt}\left[\det(A(t))\right] = \sum_{i,j=1}^{n}\left(\frac{d}{dt}a_{ij}(t)\right)A_{ij},$$

where A_{ij} is the (i,j) cofactor of A. We abuse the notation and denote the (i,k) cofactor of the derivative of the flow map by J_{ik}. It follows that

$$\begin{aligned}
\frac{DJ}{Dt} &= \sum_{i,k=1}^{3}\frac{D}{Dt}\left(\frac{\partial x_i}{\partial\alpha_k}\right)J_{ik} \\
&= \sum_{i,k=1}^{3}\frac{\partial}{\partial\alpha_k}\left(\frac{Dx_i}{Dt}\right)J_{ik} && \left[\text{Exchanging order of derivative.}\right] \\
&= \sum_{i,k=1}^{3}\frac{\partial}{\partial\alpha_k}\left(u_i(\boldsymbol{\phi}(\boldsymbol{\alpha},t),t)\right)J_{ik} && \left[\text{From (2.1.1).}\right]
\end{aligned}$$

$$= \sum_{i,k=1}^{3} \left(\sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j} \frac{\partial x_j}{\partial \alpha_k} \right) J_{ik} \qquad \Big[\text{Chain rule.}\Big]$$

$$= \sum_{j=1}^{3} \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_j} \left(\sum_{k=1}^{3} \frac{\partial x_j}{\partial \alpha_k} J_{ik} \right).$$

If $i = j$, then

$$\sum_{k=1}^{3} \frac{\partial x_i}{\partial \alpha_k} J_{ik} = J.$$

Otherwise, we have that

$$\sum_{k=1}^{3} \frac{\partial x_j}{\partial \alpha_k} J_{ik} = 0,$$

since we are taking the determinant of a matrix with 2 identical rows. Hence,

$$\frac{DJ}{Dt} = \sum_{j=1}^{3} \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_j} J \delta_{ij} = J \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j} = J \nabla \cdot \boldsymbol{u}.$$

■

2.1.3 Pathlines, streamlines and streaklines

In kinematics, we assume *a-priori* knowledge of the fluid motion through an Eulerian velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ or a Lagrangian coordinate $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{\alpha}, t)$, irrespective of the cause of the motion. Understanding the field lines, *i.e.* certain level sets associated to the underlying velocity field $\boldsymbol{u} = (u_1, u_2, u_3)$, can be useful in fluid dynamics.

1. At any fixed time t, **streamline** is a curve which is everywhere tangent to the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$, *i.e.* an integral curve of $\boldsymbol{u}(\boldsymbol{x}, t)$ for t fixed. Mathematically, the streamline $(x(s), y(s), z(s))$ satisfies the system of equations

$$\begin{aligned} \frac{dx}{ds} &= u_1 \left(x(s), y(s), z(s), t \right) \\ \frac{dy}{ds} &= u_2 \left(x(s), y(s), z(s), t \right) \\ \frac{dz}{ds} &= u_3 \left(x(s), y(s), z(s), t \right) \end{aligned}$$

   where s is the parameterisation variable of the streamline. *The shape of the streamlines is obtained by eliminating the parameter s.*

2. A **particle path** consists of points occupied by a given fluid particle as it moves in time. Mathematically, the particle path $(x(t), y(t), z(t))$ is the solution to the initial-value problem

$$\begin{aligned} \frac{dx}{dt} &= u_1 \left(x(t), y(t), z(t), t \right), \quad x(0) = x_0, \\ \frac{dy}{dt} &= u_2 \left(x(t), y(t), z(t), t \right), \quad y(0) = y_0, \end{aligned}$$

$$\frac{dz}{dt} = u_3\left(x(t), y(t), z(t), t\right), \quad z(0) = z_0,$$

where (x_0, y_0, z_0) is the particle's initial location. Physically, particle paths are trajectories of fluid particles. *The shape of the particle paths is obtained by eliminating* t.

3. A **streakline** is a curve which, at some time t, consists of the current locations of fluid particles all of which were at a given location x_0 at some earlier time. Mathematically, the streakline consists of points $\boldsymbol{x}(t)$ satisfying

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{u}(\boldsymbol{x}(t), t) \text{ and } \boldsymbol{x}(\tau) = \boldsymbol{x}_0 \text{ for some } \tau \leq t.$$

Note that the streamlines, pathlines and streaklines coincide in the case of a steady flow. For a particular time t_0, along a streamline we have

$$\frac{1}{\lambda(s)}\frac{d\boldsymbol{x}}{ds} = \boldsymbol{u}(\boldsymbol{x}, t_0),$$

where $\lambda(s)$ is an arbitrary function. Since $\boldsymbol{u}$ does not depend on t, we must have

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{u}(\boldsymbol{x}, t_0) = \frac{1}{\lambda(s)}\frac{d\boldsymbol{x}}{ds}.$$

The definition of streamlines and pathlines coincides if $\lambda(s)$ is chosen such that

$$\frac{ds}{dt} = \frac{1}{\lambda s} \iff t = \int_{s_0}^{s} \lambda(s)\, ds.$$

In other words, time is a specific parameterisation of the streamline. In what follows, we will show that these curves are very different in general by considering simple two-dimensional flows.

Example 2.1.5. Consider a two-dimensional unsteady flow $\boldsymbol{u}(\boldsymbol{x}, t) = (yt, 1)$. Streamlines are curves $\boldsymbol{x}(s; \boldsymbol{x}_0, t) = (x(s), y(s))$ with $\boldsymbol{x}_0 = (x(0), y(0))$ and t fixed, satisfying

$$\frac{dx}{ds} = yt, \quad \frac{dy}{ds} = 1.$$

Solving the second equation yields $y(s) = y_0 + s$, while solving the first equation yields

$$\begin{aligned}\frac{dx}{ds} &= yt = y_0 t + st\\ x(s) &= C + y_0 ts + \frac{ts^2}{2} = x_0 + y_0 ts + \frac{ts^2}{2}.\end{aligned}$$

Eliminating the parameter s, we see that

$$x = x_0 + y_0 t(y - y_0) + \frac{1}{2}t(y - y_0)^2,$$

i.e. the streamline at a fixed time t is a parabola for any given $\boldsymbol{x}_0$. Pathlines are curves $\boldsymbol{x}(t; \boldsymbol{x}_0) = (x(t), y(t))$ with $\boldsymbol{x}_0 = (x(0), y(0))$ fixed but t varying now. They satisfy

$$\frac{dx}{dt} = yt, \quad \frac{dy}{dt} = 1.$$

Solving the second equation yields $y(t) = y_0 + t$, while solving the first equation yields

$$\begin{aligned} \frac{dx}{dt} &= yt = y_0 t + t^2 \\ x(t) &= C + \frac{y_0 t^2}{2} + \frac{t^3}{3} = x_0 + \frac{y_0 t^2}{2} + \frac{t^3}{3}. \end{aligned}$$

Eliminating t, we see that

$$x = x_0 + \frac{1}{2} y_0 (y - y_0)^2 + \frac{1}{3}(y - y_0)^3,$$

i.e. the pathline is a cubic curve for any given $\boldsymbol{x}_0$. Streaklines are curves $\boldsymbol{x}(t; \boldsymbol{x}_0, \tau)$ with $\boldsymbol{x}(\tau) = \boldsymbol{x}_0$ fixed but t varying. They satisfy the same differential equations as pathlines but the initial condition is imposed at $t = \tau$ rather than at $t = 0$. Solving the differential equation for $y(t)$ yields $y(t) = y_0 + t - \tau$, while solving the differential equation for $x(t)$ yields

$$\begin{aligned} \frac{dx}{dt} &= yt = y_0 t + t^2 - \tau t \\ x(t) &= C + \frac{y_0 t^2}{2} + \frac{t^3}{3} - \frac{\tau t^2}{2} \\ &= \left(x_0 - \frac{y_0 \tau^2}{2} - \frac{\tau^3}{3} + \frac{\tau^3}{2} \right) + \frac{y_0 t^2}{2} + \frac{t^3}{3} - \frac{\tau t^2}{2} \\ &= x_0 + \left(\frac{1}{2} y_0 t^2 + \frac{1}{3} t^3 - \frac{\tau}{2} t^2 \right) - \left(\frac{y_0 \tau^2}{2} - \frac{\tau^3}{6} \right). \end{aligned}$$

Eliminating τ, we obtain the equation of the streaklines at time $t > \tau$:

$$x = -\frac{1}{6} y^3 + \frac{1}{2} t y^2 + \frac{1}{2} y_0^2 y + \left(x_0 - \frac{1}{2} t y_0^2 - \frac{1}{3} y_0^3 \right).$$

Example 2.1.6. Consider the unsteady flow

$$u = u_0, \quad v = kt, \quad w = 0,$$

where u_0 and k are positive constants. Since the velocity field is independent of the spatial variables x, y, z, the streamlines are straight lines. To see this, solving

$$\frac{dx}{ds} = u_0, \quad \frac{dy}{ds} = kt, \quad \frac{dz}{ds} = 0,$$

where s is the parameterisation parameter of the streamlines we obtain

$$\begin{aligned} x(s) &= u_0 s + C_1 \\ y(s) &= kst + C_2 \\ z(s) &= s + C_3. \end{aligned}$$

For any fixed time $t \geq 0$, these equations are the parametric form of the equation of a line in the three-dimensional Cartesian plane. We can find the particle path of a given fluid particle by solving

$$\frac{dx}{dt} = u_0, \quad \frac{dy}{dt} = kt, \quad \frac{dz}{dt} = 0.$$

This has general solution of the form

$$x(t) = u_0 t + x_0$$
$$y(t) = \frac{kt^2}{2} + y_0$$
$$z(t) = t + z_0$$

where (x_0, y_0, z_0) is the initial position of the fluid particle. In particular, any fluid particle follows a parabolic path as time proceeds since the y-component of the particle path is quadratic in t.

Example 2.1.7. Consider the two-dimensional steady flow

$$u = \lambda x, \quad v = -\lambda y.$$

where α is some positive constants. Given a fluid particle initially at $\boldsymbol{\alpha} = (a, b)$, its particle path can be found as follows:

$$\frac{dx}{dt} = \lambda x, \quad x(0) = a \implies x(t) = ae^{\lambda t}$$
$$\frac{dy}{dt} = -\lambda y, \quad y(0) = b \implies y(t) = be^{-\lambda t}$$

Since $xy = ab$, the particle paths are *hyperbolas* and the point $x = y = 0$ where the velocity is zero is called a *stagnation point*. Although the Eulerian velocity field is steady, the Lagrangian velocity field is unsteady. Indeed, it follows from (2.1.1) that

$$\boldsymbol{u}(\boldsymbol{\alpha}, t) = (\lambda ae^{\lambda t}, -\lambda be^{-\lambda t}).$$

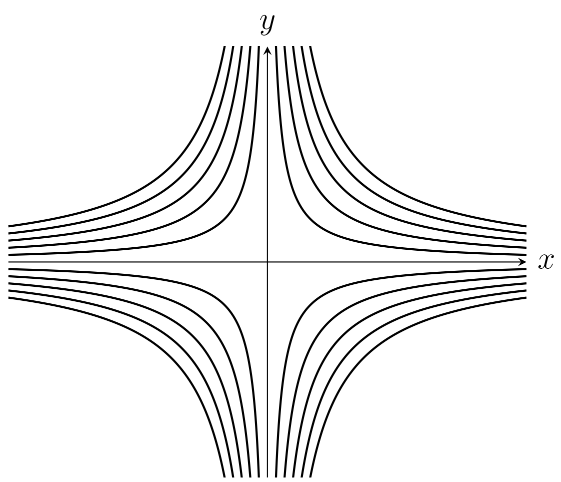

Figure 2.2: Hyperbolic particle paths of a stagnation-point flow.

2.2 Conservation Equations

2.2.1 Continuity equation

We assume that there is a well-defined function $\rho(\boldsymbol{x}, t)$, called the fluid density, such that the mass of any parcel of fluid Ω_t is

$$m\left(\Omega_t\right) = \int_{\Omega_t} \rho(\boldsymbol{x}, t)\, dV_{\boldsymbol{x}}.$$

We want to derive a PDE for the density, assuming that mass is neither created nor destroyed; this is known as the law of conservation of mass. More precisely, consider an initial arbitrary parcel of fluid $\Omega_0 \equiv \Omega(0)$ at time $t = 0$. Its mass is given by

$$m\left(\Omega_0\right) = \int_{\Omega_0} \rho(\boldsymbol{x}, 0)\, dV_{\boldsymbol{x}}.$$

Conservation of mass then asserts that

$$m\left(\Omega_0\right) = m\left(\Omega_t\right) \quad \text{for all } t \geq 0. \tag{2.2.1}$$

Differentiating (2.2.1) with respect to time, we obtain

$$0 = \frac{d}{dt}\left(\int_{\Omega_0} \rho(\boldsymbol{x}, 0)\, dV_{\boldsymbol{x}}\right) = \frac{d}{dt}\left(\int_{\Omega_t} \rho(\boldsymbol{x}, t)\, dV_{\boldsymbol{x}}\right). \tag{2.2.2}$$

Unfortunately, we cannot apply Leibniz's rule since the domain of integration is time-dependent. To do this, we make a change of variables $\boldsymbol{x} = \boldsymbol{\phi}\left(\boldsymbol{\alpha}, t\right)$ that maps Ω_t to the initial region Ω_0. Performing the change of variables $\boldsymbol{x} = \boldsymbol{\phi}\left(\boldsymbol{\alpha}, t\right)$ in (2.2.2) yields

$$\begin{aligned}
0 &= \frac{d}{dt}\int_{\Omega_t} \rho(\boldsymbol{x}, t)\, dV_{\boldsymbol{x}} \\
&= \frac{d}{dt}\int_{\Omega_0} \rho(\boldsymbol{\phi}\left(\boldsymbol{\alpha}, t\right))\left|J(\boldsymbol{\alpha}, t)\right|\, dV_{\boldsymbol{\alpha}} \\
&= \int_{\Omega_0} \frac{D}{Dt}\Big[\rho(\boldsymbol{\phi}\left(\boldsymbol{\alpha}, t\right))\left|J(\boldsymbol{\alpha}, t)\right|\Big]\, dV_{\boldsymbol{\alpha}} && \Big[\text{Leibniz's rule.}\Big] \\
&= \int_{\Omega_0} \left[\frac{D\rho}{Dt}(\boldsymbol{\phi}\left(\boldsymbol{\alpha}, t\right))\left|J(\boldsymbol{\alpha}, t)\right| + \rho(\boldsymbol{\phi}\left(\boldsymbol{\alpha}, t\right))\frac{D|J|}{Dt}(\boldsymbol{\alpha}, t)\right] dV_{\boldsymbol{\alpha}} && \Big[\text{Product rule.}\Big] \\
&= \int_{\Omega_0} \left[\frac{D\rho}{Dt}|J| + \rho\nabla\cdot\boldsymbol{u}|J|\right]\Bigg|_{\boldsymbol{\phi}(\boldsymbol{\alpha},t)} dV_{\boldsymbol{\alpha}} && \Big[\text{From Lemma (2.1.2).}\Big] \\
&= \int_{\Omega_0} \left[\left(\frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{u}\right)|J|\right]\Bigg|_{\boldsymbol{\phi}(\boldsymbol{\alpha},t)} dV_{\boldsymbol{\alpha}} \\
&= \int_{\Omega_t} \left[\frac{D\rho}{Dt}(\boldsymbol{x}, t) + \rho\nabla\cdot\boldsymbol{u}(\boldsymbol{x}, t)\right] dV_{\boldsymbol{x}}.
\end{aligned}$$

Since Ω_t was arbitrary, the integrand must be zero and we obtain the continuity equation in the Eulerian coordinates:

$$\boxed{\frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{u} = 0.} \tag{2.2.3}$$

This can be rewritten as

$$\boxed{\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{u} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0.} \tag{2.2.4}$$

In Lagrangian coordinates, the conservation of mass takes the form

$$\boxed{\frac{D}{Dt}(J\rho) = 0} \tag{2.2.5}$$

which can be seen from the Euler's identity (2.1.2).

Example 2.2.1. Consider the one-dimensional Eulerian velocity field $u(x,t) = \dfrac{2x}{1+t}$. From Example (2.1.2), the flow map is given by $\phi(\alpha, t) = \alpha(1+t)^2$, with Jacobian

$$J = \frac{\partial x}{\partial \alpha} = (1+t)^2.$$

Suppose the density satisfies $\rho(\alpha, 0) = \alpha$ and mass is conserved. From (2.2.5), we obtain

$$J\rho = (1+t)^2 \rho = C \implies \rho(\alpha, t) = \frac{C}{(1+t)^2} = \frac{\alpha}{(1+t)^2}.$$

As a function of x and t, the density takes the form

$$\rho(x,t) = \frac{x}{(1+t)^4}.$$

For completeness, we verify that $\rho(x,t)$ and $u(x,t)$ satisfies the Eulerian continuity equation (2.2.4):

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) &= -\frac{4x}{(1+t)^5} + \frac{\partial}{\partial x}\left[\frac{2x^2}{(1+t)^5}\right] \\ &= -\frac{4x}{(1+t)^5} + \frac{4x}{(1+t)^5} = 0.\end{aligned}$$

Remark 2.2.2. We provide another derivation of the continuity equation using the idea of flux. Consider a fluid whose density is equal to $\rho(\boldsymbol{x}, t)$ and whose velocity is given by $\boldsymbol{u}(\boldsymbol{x}, t)$. Let $M(t)$ be the mass of fluid in a region $\mathcal{D} \subset \mathbb{R}^3$ at time t. Then

$$M(t) = \int_{\mathcal{D}} \rho(\boldsymbol{x}, t)\, dV_{\boldsymbol{x}}.$$

By conservation of mass, the rate at which mass of fluid flows out of $\mathcal{D}$ must equal the flux of mass across the boundary $\partial\mathcal{D}$, *i.e.*

$$-\frac{dM(t)}{dt} = \int_{\partial\mathcal{D}} \rho(\boldsymbol{x}, t)\boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{n}\, dS_{\boldsymbol{x}},$$

where $\rho(\boldsymbol{x},t)\boldsymbol{u}(\boldsymbol{x},t)$ is the flux of mass and $\boldsymbol{n}$ is the outward unit normal of $\partial\mathcal{D}$. Using the divergence theorem and Leibniz's rule, we obtain

$$-\int_{\mathcal{D}} \frac{\partial \rho}{\partial t}\, dV_{\boldsymbol{x}} = \int_{\mathcal{D}} \nabla\cdot(\rho\boldsymbol{u})\, dV_{\boldsymbol{x}}$$
$$\int_{\mathcal{D}} \left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u})\right) dV_{\boldsymbol{x}} = 0.$$

Definition 2.2.3. A flow is **incompressible** if $\nabla\cdot\boldsymbol{u} \equiv 0$. (This is not a property of a fluid.) For such a flow, the fluid density satisfies

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u}\cdot\nabla\rho = \frac{D\rho}{Dt} = 0.$$

This does not mean that the density is constant everywhere, it simply says that if we start with a chunk of fluid with density ρ, then it remains unchanged along that chunk of fluid. However, a fluid of constant density without mass addition must be incompressible.

For an incompressible flow, an initially homogeneous fluid remains homogeneous. More precisely, if in addition to incompressibility, the initial density is constant, *i.e.* $\rho(\boldsymbol{x},0) = \rho_0$, then $\rho(\boldsymbol{x},t) = \rho_0$ for all $\boldsymbol{x}, t$. Another simple consequence of incompressibility is that the volume of a chunk of fluid does not change. If the flow is incompressible, it follows from the Euler's identity (2.1.2) that

$$\frac{DJ}{Dt} = 0 \implies J(\boldsymbol{\alpha},t) = C.$$

Since $J(\boldsymbol{\alpha},0) = 1$, we must have $J(\boldsymbol{\alpha},t) \equiv 1$ for all $t \geq 0$, *i.e.* J is the determinant of the identity matrix. For a fluid $\Omega(0)$ occupying $\Omega(t)$ at time t, the volume of $\Omega(t)$ is

$$|\Omega(t)| = \int_{\Omega_t} dV_{\boldsymbol{x}} = \int_{\Omega_0} |J(\boldsymbol{\alpha},t)|\, dV_{\boldsymbol{\alpha}} = \int_{\Omega_0} dV_{\alpha} = |\Omega(0)|.$$

2.2.2 Reynolds transport theorem

By a similar argument to the derivation of the continuity equation, we can derive a more general result, known as the **Reynolds Transport Theorem**:

Theorem 2.2.4. *If $F(\boldsymbol{x},t)$ is a scalar function that is defined in the fluid, then*

$$\frac{d}{dt}\int_{\Omega_t} F(\boldsymbol{x},t)\, dV_{\boldsymbol{x}} = \int_{\Omega_t} \left(\frac{\partial F}{\partial t} + \nabla\cdot(F\boldsymbol{u})\right) dV_{\boldsymbol{x}}.$$

A special case and practically more useful of the Reynolds Transport Theorem is as follows: Given a fluid density $\rho(\boldsymbol{x},t)$,

$$\frac{d}{dt}\int_{\Omega_t} \rho(\boldsymbol{x},t)F(\boldsymbol{x},t)\, dV_{\boldsymbol{x}} = \int_{\Omega_t} \rho\frac{DF}{Dt}\, dV_{\boldsymbol{x}}. \qquad \text{(RTT)}$$

Indeed, a simple application of product rule gives

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t}\rho F\,dV_{\boldsymbol{x}} &= \int_{\Omega_t}\left(\frac{\partial(\rho F)}{\partial t}+\nabla\cdot(\rho F\boldsymbol{u})\right)dV_{\boldsymbol{x}}\\ &= \int_{\Omega_t}\left(F\frac{\partial \rho}{\partial t}+\rho\frac{\partial F}{\partial t}+\rho\boldsymbol{u}\cdot\nabla F+F\nabla\cdot(\rho\boldsymbol{u})\right)dV_{\boldsymbol{x}}\\ &= \int_{\Omega_t}F\left(\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{u})\right)+\rho\left(\frac{\partial F}{\partial t}+\boldsymbol{u}\cdot\nabla F\right)dV_{\boldsymbol{x}}\\ &= \int_{\Omega_t}\rho\frac{DF}{Dt}\,dV_{\boldsymbol{x}},\end{aligned}$$

where the last equality follows from the Eulerian continuity equation (2.2.4).

2.2.3 Conservation of linear momentum

Observe that the continuity equation is a single equation with four unknowns functions ρ and the three components of $\boldsymbol{u}$. We require three more equations and these are provided by Newton's laws of motion for the fluid. Recall that Newton's second law of motion states that the time rate of change of linear momentum of a particle equals to the sum of forces acting on the particle. For a blob of fluid occupying $\Omega(t)$ at time t, its linear momentum is given by

$$\frac{d}{dt}\int_{\Omega_t}\rho(\boldsymbol{x},t)\boldsymbol{u}(\boldsymbol{x},t)\,dV_{\boldsymbol{x}}. \tag{2.2.6}$$

We distinguish two types of forces acting on the blob of fluid Ω_t:

1. Body forces such as gravitational or electromagnetic force can be regarded as forces acting throughout the volume. We denote by $\boldsymbol{F}_b(\boldsymbol{x},t)$ the external force per unit mass.

2. Surface forces such as pressure or viscous stresses can be regarded as forces acting on the volume through its boundary. We denote by $\boldsymbol{t}(\boldsymbol{x},t)$ the force per unit area exerted at $\boldsymbol{x}$ on fluid inside Ω_t by the fluid outside Ω_t. $\boldsymbol{t}(\boldsymbol{x},t)$ is sometimes called the **traction** or stress vector.

Including these two forces we obtain

$$\int_{\Omega_t}\rho(\boldsymbol{x},t)\frac{D\boldsymbol{u}}{Dt}(\boldsymbol{x},t)\,dV_{\boldsymbol{x}} = \int_{\Omega_t}\underbrace{\rho(\boldsymbol{x},t)\boldsymbol{F}_b(\boldsymbol{x},t)}_{\text{volume force density}}\,dV_{\boldsymbol{x}}+\int_{\partial\Omega_t}\underbrace{\boldsymbol{t}(\boldsymbol{x},t)}_{\text{surface force density}}\,dS_{\boldsymbol{x}}. \tag{2.2.7}$$

Theorem 2.2.5 (Principle of local stress equilibrium). *Let ℓ be some characteristic length for a sequence of regions around a point $\boldsymbol{x}$. Then*

$$\lim_{\ell\to 0}\frac{1}{\ell^2}\int_{\partial\Omega_\ell(t)}\boldsymbol{t}(\boldsymbol{x},t)\,dS_{\boldsymbol{x}}=0,$$

i.e. there is no net force due to fluids around it.

Proof. Suppose $\{\Omega_\ell(t)\}$ is a sequence of regions of a given shape around a point $\boldsymbol{x}$ with characteristic length l, say the cubic root of its volume, such that the region becomes smaller as $\ell \longrightarrow 0$. The volume and surface area of $\Omega_\ell(t)$ are proportional to ℓ^3 and ℓ^2 respectively, with the proportionality constants depending only on the shape. Consider (2.2.7) over the region $\Omega_\ell(t)$. Dividing each side by ℓ^2 gives

$$\frac{1}{\ell^2}\int_{\Omega_\ell(t)} \rho\frac{D\boldsymbol{u}}{Dt}\,dV_{\boldsymbol{x}} = \frac{1}{\ell^2}\int_{\Omega_\ell(t)} \rho\boldsymbol{F}_b\,dV_{\boldsymbol{x}} + \frac{1}{\ell^2}\int_{\partial\Omega_\ell(t)} \boldsymbol{t}\,dS_{\boldsymbol{x}}.$$

Assuming all the integrands are bounded. If we allow $\Omega_\ell(t)$ to shrink to the point $\boldsymbol{x}$ while preserving its shape, we see that the volume integrals converge to 0 as $\ell \longrightarrow 0$ and the theorem is proved.

■

Corollary 2.2.6. *Below are a few consequences of the principle of local stress equilibrium:*

(a) $\boldsymbol{t}$ *cannot be a function of* $\boldsymbol{x}$ *and* t *only. We assume that it depends also on* $\boldsymbol{n}$*, the outward unit normal of* $\partial\Omega_t$ *at the point* $\boldsymbol{x}$*, i.e.* $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})$*.*

(b) $\boldsymbol{t}$ *must be odd with respect to* $\boldsymbol{n}$*, i.e.* $\boldsymbol{t}(\boldsymbol{x}, t, -\boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})$*. This is equivalent to Newton's third law of motion, which says that the stress vector acting on opposite sides of the same surface is equal in magnitude and opposite in direction.*

(c) Most importantly, $\boldsymbol{t}$ *depends linearly on* $\boldsymbol{n}$*, and it has an explicit representation formula*

$$\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) = \boldsymbol{n} \cdot \underline{\underline{T}}(\boldsymbol{x}, t),$$

where $\underline{\underline{T}}(\boldsymbol{x}, t)$ *is the second-order tensor called the* ***Cauchy stress tensor****.*

Proof. To prove this, consider a tetrahedron with three faces A_1, A_2, A_3 oriented in the coordinate planes, and with an infinitesimal area dA oriented in an arbitrary direction specified by $\boldsymbol{n}$. Shrinking the tetrahedron and assuming that the stress vector is constant in the shrinking tetrahedron, we obtain from the principle of local stress equilibrium

$$0 = \lim_{dA\to 0}\frac{1}{dA}\Big[\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n})dA + \boldsymbol{t}(\boldsymbol{x}, -\boldsymbol{e}_1)dA_1 + \boldsymbol{t}(\boldsymbol{x}, -\boldsymbol{e}_2)dA_2 + \boldsymbol{t}(\boldsymbol{x}, -\boldsymbol{e}_3)dA_3\Big]. \tag{2.2.8}$$

We claim that the areas of other faces dA_j are $n_j dA$. Indeed, invoking the generalised divergence theorem with the identity tensor and abusing the notation for $\boldsymbol{n}$ yield

$$\begin{aligned} 0 &= \int_{A\cup A_1\cup A_2\cup A_3} \boldsymbol{n}\cdot\underline{\underline{I}}\,dS \\ &= \int_{A_1} -\boldsymbol{e}_1\,dS + \int_{A_2} -\boldsymbol{e}_2\,dS + \int_{A_3} -\boldsymbol{e}_3\,dS + \int_A \boldsymbol{n}\,dS \\ &= -dA_1\boldsymbol{e}_1 - dA_2\boldsymbol{e}_2 - dA_3\boldsymbol{e}_3 + dA\boldsymbol{n}, \end{aligned}$$

which then implies

$$n_j dA = (\boldsymbol{n}\cdot\boldsymbol{e}_j)dA = \boldsymbol{e}_j\cdot\boldsymbol{e}_k dA_k = dA_j.$$

This together with the fact that $\boldsymbol{t}$ is odd in $\boldsymbol{n}$ reduces (2.2.8) to

$$0 = \lim_{dA\to 0}\frac{1}{dA}\Big[\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n})dA - \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{e}_1)n_1 dA - \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{e}_2)n_2 dA - \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{e}_2)n_2 dA\Big]$$

$$\boldsymbol{t}(\boldsymbol{x},\boldsymbol{n}) = \boldsymbol{t}(\boldsymbol{x},\boldsymbol{e}_1)n_1 + \boldsymbol{t}(\boldsymbol{x},\boldsymbol{e}_2)n_2 + \boldsymbol{t}(\boldsymbol{x},\boldsymbol{e}_3)n_3 = \boldsymbol{n}\cdot\underline{\underline{T}}(\boldsymbol{x}),$$

where

$$\underline{\underline{T}}(\boldsymbol{x}) = \begin{bmatrix}\boldsymbol{t}(\boldsymbol{x},\boldsymbol{e}_1)\\ \boldsymbol{t}(\boldsymbol{x},\boldsymbol{e}_2)\\ \boldsymbol{t}(\boldsymbol{x},\boldsymbol{e}_3)\end{bmatrix}.$$

Here, T_{ij} is the component of the surface force per unit area in the jth direction on a surface whose normal is pointing in the ith coordinate direction. T_{ii} are the normal stresses and $T_{ij}, i \neq j$ are the shear stresses.

■

Finally, substituting $\boldsymbol{t} = \boldsymbol{n}\cdot\underline{\underline{T}}$ and applying the generalised divergence theorem on the surface integral in (2.2.7) results in

$$\begin{aligned}\int_{\Omega_t} \rho(\boldsymbol{x},t)\frac{D\boldsymbol{u}}{Dt}(\boldsymbol{x},t)\,dV_{\boldsymbol{x}} &= \int_{\Omega_t} \rho(\boldsymbol{x},t)\boldsymbol{F}_b(\boldsymbol{x},t)\,dV_{\boldsymbol{x}} + \int_{\partial\Omega_t} \boldsymbol{n}\cdot\underline{\underline{T}}(\boldsymbol{x},t)\,dS_{\boldsymbol{x}}\\ &= \int_{\Omega_t}\Big[\rho(\boldsymbol{x},t)\boldsymbol{F}_b(\boldsymbol{x},t) + \nabla\cdot\underline{\underline{T}}(\boldsymbol{x},t)\Big]\,dV_{\boldsymbol{x}}.\end{aligned}$$

Since Ω_t was arbitrary, we must have

$$\rho\left(\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = \rho\boldsymbol{F}_b + \nabla\cdot\underline{\underline{T}}. \tag{2.2.9}$$

This is sometimes known as the **Cauchy momentum equation**. Together with the continuity equation, we now have 4 equations but 13 unknowns $\rho, \boldsymbol{u}, \underline{\underline{T}}$.

2.2.4 Conservation of angular momentum

It turns out that $\underline{\underline{T}}$ is symmetric according to the law of conservation of angular momentum, under certain assumptions. Assuming that there are no microscopic scale contributions to the torque, we have

$$\frac{d}{dt}\int_{\Omega(t)} \boldsymbol{x}\times(\rho\boldsymbol{u})\;dV_{\boldsymbol{x}} = \text{sum of torques acting on }\Omega(t) \tag{2.2.10a}$$

$$= \int_{\Omega(t)} \boldsymbol{x}\times(\rho\boldsymbol{F}_b)\;dV_{\boldsymbol{x}} + \int_{\partial\Omega(t)} \boldsymbol{x}\times(\boldsymbol{n}\cdot\underline{\underline{T}})\,dS_{\boldsymbol{x}}. \tag{2.2.10b}$$

We now prove two useful identities that will allow us to invoke the law of conservation of linear momentum (2.2.9) and greatly simplify (2.2.10).

Lemma 2.2.7. *Let $\underline{\underline{T}}$ be a rank 2 tensor, $\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{n}$ rank 1 tensors and ε the permutation symbol.*

(a) For a region $\mathcal{D}\subset\mathbb{R}^3$ with boundary $\partial\mathcal{D}$, the following identity holds:

$$\int_{\partial\mathcal{D}} \boldsymbol{x}\times\left(\boldsymbol{n}\cdot\underline{\underline{T}}\right)\;dS_{\boldsymbol{x}} = \int_{\mathcal{D}}\left[\boldsymbol{x}\times\left(\nabla\cdot\underline{\underline{T}}\right) + \varepsilon : \underline{\underline{T}}\right]\;dV_{\boldsymbol{x}}. \tag{2.2.11}$$

(b) For a moving region Ω_t *and density* ρ*, the following identity holds:*

$$\frac{d}{dt}\int_{\Omega_t} \boldsymbol{x} \times (\rho \boldsymbol{u})\, dV_{\boldsymbol{x}} = \int_{\Omega_t} \boldsymbol{x} \times \left(\rho \frac{D\boldsymbol{u}}{Dt}\right) dV_{\boldsymbol{x}}. \tag{2.2.12}$$

Proof. For part (a), we first write out the integrand of the integral over $\mathcal{D}$:

$$\begin{aligned} \boldsymbol{x} \times \left(\nabla \cdot \underline{\underline{T}}\right) + \varepsilon : \underline{\underline{T}} &= (x_i \boldsymbol{e}_i) \times (\partial_j T_{jk} \boldsymbol{e}_k) + \varepsilon_{ijk} T_{jk} \boldsymbol{e}_i \\ &= x_i \partial_j T_{jk} \varepsilon_{ikm} \boldsymbol{e}_m + \varepsilon_{ijk} T_{jk} \boldsymbol{e}_i \\ &= x_i \partial_j T_{jk} \varepsilon_{ikm} \boldsymbol{e}_m + \varepsilon_{mjk} T_{jk} \boldsymbol{e}_m. \end{aligned}$$

Applying the generalised divergence theorem onto the left integral yields

$$\begin{aligned} \int_{\partial\mathcal{D}} \boldsymbol{x} \times \left(\boldsymbol{n} \cdot \underline{\underline{T}}\right) dS_{\boldsymbol{x}} &= \int_{\partial\mathcal{D}} (x_i \boldsymbol{e}_i) \times (n_j T_{jk} \boldsymbol{e}_k)\, dS_{\boldsymbol{x}} \\ &= \int_{\partial\mathcal{D}} n_j x_i T_{jk} \varepsilon_{ikm} \boldsymbol{e}_m \, dS_{\boldsymbol{x}} \\ &= \int_{\mathcal{D}} \partial_j (x_i T_{jk})\, \varepsilon_{ikm} \boldsymbol{e}_m \, dV_{\boldsymbol{x}} \\ &= \int_{\mathcal{D}} x_i \partial_j T_{jk} \varepsilon_{ikm} \boldsymbol{e}_m + T_{jk} \partial_j x_i \varepsilon_{ikm} \boldsymbol{e}_m \, dV_{\boldsymbol{x}} \\ &= \int_{\mathcal{D}} x_i \partial_j T_{jk} \varepsilon_{ikm} \boldsymbol{e}_m + T_{jk} \varepsilon_{jkm} \boldsymbol{e}_m \, dV_{\boldsymbol{x}} \\ &= \int_{\mathcal{D}} x_i \partial_j T_{jk} \varepsilon_{ikm} \boldsymbol{e}_m + T_{jk} \varepsilon_{mjk} \boldsymbol{e}_m \, dV_{\boldsymbol{x}} \\ &= \int_{\mathcal{D}} \left[\boldsymbol{x} \times \left(\nabla \cdot \underline{\underline{T}}\right) + \varepsilon : \underline{\underline{T}}\right] dV_{\boldsymbol{x}}. \end{aligned}$$

Part (b) is essentially an application of the Reynolds Transport Theorem (RTT). First, observe that we can factor out the scalar function ρ on each side, so it suffices to show that

$$\frac{d}{dt}\int_{\Omega_t} \rho\, (\boldsymbol{x} \times \boldsymbol{u})\, dV_{\boldsymbol{x}} = \int_{\Omega_t} \rho \left(\boldsymbol{x} \times \frac{D\boldsymbol{u}}{Dt}\right) dV_{\boldsymbol{x}}.$$

Applying the Reynolds Transport Theorem (RTT) to the left integral gives

$$\frac{d}{dt}\int_{\Omega_t} \rho\, (\boldsymbol{x} \times \boldsymbol{u})\, dV_{\boldsymbol{x}} = \int_{\Omega_t} \rho \frac{D}{Dt} (\boldsymbol{x} \times \boldsymbol{u})\, dV_{\boldsymbol{x}}.$$

Thus, we only need to show that

$$\frac{D}{Dt}(\boldsymbol{x} \times \boldsymbol{u}) = \boldsymbol{x} \times \frac{D\boldsymbol{u}}{Dt}.$$

From the definition of material derivative,

$$\begin{aligned} \frac{D}{Dt}(\boldsymbol{x} \times \boldsymbol{u}) = \frac{D}{Dt}\Big[(x_i \boldsymbol{e}_i) \times (u_j \boldsymbol{e}_j)\Big] &= \frac{D}{Dt}(x_i u_j \varepsilon_{ijk} \boldsymbol{e}_k) \\ &= \varepsilon_{ijk} \frac{\partial}{\partial t}(x_i u_j \boldsymbol{e}_k) + \boldsymbol{u} \cdot \nabla (x_i u_j \varepsilon_{ijk} \boldsymbol{e}_k)\,, \end{aligned}$$

and expanding the second term yields

$$\begin{aligned}\boldsymbol{u} \cdot \nabla\left(x_i u_j \varepsilon_{ijk} \boldsymbol{e}_k\right) &= \left(u_m \boldsymbol{e}_m\right) \cdot \left(\partial_n \left(x_i u_j\right) \varepsilon_{ijk} \boldsymbol{e}_n \boldsymbol{e}_k\right) \\ &= \varepsilon_{ijk} u_m \partial_n \left(x_i u_j\right) \delta_{mn} \boldsymbol{e}_k \\ &= \varepsilon_{ijk} u_m \partial_m \left(x_i u_j\right) \boldsymbol{e}_k \\ &= \varepsilon_{ijk} u_m \delta_{im} u_j \boldsymbol{e}_k + \varepsilon_{ijk} u_m x_i \partial_m u_j \boldsymbol{e}_k \\ &= \varepsilon_{ijk} u_i u_j \boldsymbol{e}_k + \varepsilon_{ijk} x_i u_m \partial_m u_j \boldsymbol{e}_k.\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{D}{Dt}\left(\boldsymbol{x} \times \boldsymbol{u}\right) &= \varepsilon_{ijk} \frac{\partial}{\partial t}\left(x_i u_j \boldsymbol{e}_k\right) + \boldsymbol{u} \cdot \nabla\left(x_i u_j \varepsilon_{ijk} \boldsymbol{e}_k\right) \\ &= \varepsilon_{ijk} x_i \partial_t u_j \boldsymbol{e}_k + \Big[\varepsilon_{ijk} u_i u_j \boldsymbol{e}_k + \varepsilon_{ijk} x_i u_m \partial_m u_j \boldsymbol{e}_k\Big] \\ &= \varepsilon_{ijk} u_i u_j \boldsymbol{e}_k + x_i \Big[\partial_t u_j + u_m \partial_m u_j\Big] \varepsilon_{ijk} \boldsymbol{e}_k \\ &= \boldsymbol{u} \times \boldsymbol{u} + \boldsymbol{x} \times \frac{D\boldsymbol{u}}{Dt}.\end{aligned}$$

The desired statement follows since $\boldsymbol{u} \times \boldsymbol{u} = \boldsymbol{0}$.

■

It follows from Lemma 2.2.7 and the Cauchy's equation of motion (2.2.9) that

$$\begin{aligned}\int_{\Omega_t} \boldsymbol{x} \times \left(\rho \frac{D\boldsymbol{u}}{Dt}\right) dV_{\boldsymbol{x}} &= \int_{\Omega_t} \boldsymbol{x} \times \left(\rho \boldsymbol{F}_b\right) dV_{\boldsymbol{x}} + \int_{\Omega_t} \boldsymbol{x} \times \left(\nabla \cdot \underline{\underline{T}}\right) dV_{\boldsymbol{x}} + \int_{\Omega_t} \varepsilon \colon \underline{\underline{T}} \, dV_{\boldsymbol{x}} \\ 0 &= \int_{\Omega_t} \varepsilon \colon \underline{\underline{T}} \, dV_{\boldsymbol{x}}.\end{aligned}$$

Since Ω_t was arbitrary, we must have

$$\varepsilon \colon \underline{\underline{T}} = \varepsilon_{ijk} T_{jk} \boldsymbol{e}_i = \boldsymbol{0}.$$

Componentwise,

$$\begin{aligned}0 &= \varepsilon_{1jk} T_{jk} = T_{23} - T_{32} &\implies T_{23} = T_{32} \\ 0 &= \varepsilon_{2jk} T_{jk} = T_{31} - T_{13} &\implies T_{13} = T_{31} \\ 0 &= \varepsilon_{3jk} T_{jk} = T_{12} - T_{21} &\implies T_{12} = T_{21}\end{aligned}$$

and so $\underline{\underline{T}}$ is a symmetric rank 2 tensor, *i.e.* $\underline{\underline{T}}$ now has 6 unknown components instead of 9. This is true for any nature of the deformable medium, as long as the net torque on the chunk of fluid is due simply to the moment of the body force per unit mass $\boldsymbol{F}_b$ and the moment of the stresses $\boldsymbol{t} = \boldsymbol{n} \cdot \underline{\underline{T}}$ on its surface.

2.2.5 Conservation of energy

The energy of fluid particles has 2 pieces:

1. Kinetic energy that is due to the motion of the velocity field. The kinetic energy density is given by
$$\frac{1}{2}\rho(\boldsymbol{x},t)\boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{u}(\boldsymbol{x},t)=\frac{1}{2}\rho(\boldsymbol{x},t)\|\boldsymbol{u}(\boldsymbol{x},t)\|^2.$$

2. Internal energy $\boldsymbol{e}(\boldsymbol{x},t)$ that is basically due to differences between molecular velocities (which we are not tracking) and the macroscopic velocity $\boldsymbol{u}$. The internal energy density is given by $\rho(\boldsymbol{x},t)e(\boldsymbol{x},t)$.

The principle of conservation of energy says that the time rate of change of energy of fluid in Ω_t is equal to the rate of work done by forces acting on fluid in Ω_t and the rate of internal energy movement across $\partial\Omega_t$. Let $\boldsymbol{q}(\boldsymbol{x},t)$ be the internal energy flux vector, then

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega_t}\left(\frac{1}{2}\rho\|\boldsymbol{u}\|^2+\rho e\right)dV_{\boldsymbol{x}}\\&=\int_{\Omega_t}\boldsymbol{u}\cdot\rho\boldsymbol{F}_b\,dV_{\boldsymbol{x}}+\int_{\partial\Omega_t}\boldsymbol{u}\cdot\boldsymbol{t}\,dS_{\boldsymbol{x}}-\int_{\partial\Omega_t}\boldsymbol{n}\cdot\boldsymbol{q}\,dS_{\boldsymbol{x}}.\end{aligned}$$

To simplify the first surface integral, observe that

$$\begin{aligned}\boldsymbol{u}\cdot\left(\boldsymbol{n}\cdot\underline{\underline{T}}\right)-\boldsymbol{n}\cdot\left(\underline{\underline{T}}\cdot\boldsymbol{u}\right)&=(u_i\boldsymbol{e}_i)\cdot((n_j\boldsymbol{e}_j)\cdot(T_{km}\boldsymbol{e}_k\boldsymbol{e}_m))-(n_i\boldsymbol{e}_i)\cdot((T_{jk}\boldsymbol{e}_j\boldsymbol{e}_k)\cdot(u_m\boldsymbol{e}_m))\\&=(u_i\boldsymbol{e}_i)\cdot(n_jT_{jm}\boldsymbol{e}_m)-(n_i\boldsymbol{e}_i)\cdot(T_{jm}u_m\boldsymbol{e}_j)\\&=u_in_jT_{jm}\delta_{im}-n_iu_mT_{jm}\delta_{ij}\\&=u_mn_jT_{jm}-n_ju_mT_{jm}=0,\end{aligned}$$

Therefore

$$\begin{aligned}\int_{\partial\Omega_t}\boldsymbol{u}\cdot\boldsymbol{t}\,dS_{\boldsymbol{x}}=\int_{\partial\Omega_t}\boldsymbol{u}\cdot\left(\boldsymbol{n}\cdot\underline{\underline{T}}\right)dS_{\boldsymbol{x}}&=\int_{\partial\Omega_t}\boldsymbol{n}\cdot\left(\underline{\underline{T}}\cdot\boldsymbol{u}\right)dS_{\boldsymbol{x}}\\&=\int_{\Omega_t}\nabla\cdot\left(\underline{\underline{T}}\cdot\boldsymbol{u}\right)dV_{\boldsymbol{x}}.\end{aligned}$$

where we used the fact that $\underline{\underline{T}}$ is symmetric. Consequently, applying Reynolds Transport Theorem (RTT) on the LHS and the generalised divergence theorem on the second surface integral yields

$$\int_{\Omega_t}\rho\frac{D}{Dt}\left(\frac{1}{2}\|\boldsymbol{u}\|+e\right)dV_{\boldsymbol{x}}=\int_{\Omega_t}\left(\rho\boldsymbol{u}\cdot\boldsymbol{F}_b+\nabla\cdot\left(\underline{\underline{T}}\cdot\boldsymbol{u}\right)-\nabla\cdot\boldsymbol{q}\right)dV_{\boldsymbol{x}}.\tag{2.2.13}$$

On the other hand, taking the dot product of $\boldsymbol{u}$ against the Cauchy's equation of motion (2.2.9) gives

$$\begin{aligned}\rho\boldsymbol{u}\cdot\frac{D\boldsymbol{u}}{Dt}&=\rho\boldsymbol{u}\cdot\boldsymbol{F}_b+\boldsymbol{u}\cdot\left(\nabla\cdot\underline{\underline{T}}\right)\\\rho\frac{D}{Dt}\left(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)&=\rho\boldsymbol{u}\cdot\boldsymbol{F}_b+\boldsymbol{u}\cdot\left(\nabla\cdot\underline{\underline{T}}\right)\end{aligned}\tag{2.2.14}$$

Substituting (2.2.14) into (2.2.13), we obtain the equation

$$\rho\frac{De}{Dt}=\nabla\cdot\left(\underline{\underline{T}}\cdot\boldsymbol{u}\right)-\boldsymbol{u}\cdot\left(\nabla\cdot\underline{\underline{T}}\right)-\nabla\cdot\boldsymbol{q}.\tag{2.2.15}$$

Applying the next lemma results in the conservation of internal energy equation:

$$\rho \frac{De}{Dt} = \underline{\underline{T}} \colon \underline{\underline{D}} - \nabla \cdot \boldsymbol{q}. \tag{2.2.16}$$

Lemma 2.2.8. $\nabla \cdot \left(\underline{\underline{T}} \cdot \boldsymbol{u}\right) - \boldsymbol{u} \cdot \left(\nabla \cdot \underline{\underline{T}}\right)$ *is a double contraction and it equals to* $\underline{\underline{T}} \colon \underline{\underline{D}}$*, where* $\underline{\underline{D}}$ *is the symmetric part of the velocity gradient tensor* $\nabla \boldsymbol{u}$*.*

Proof. We first expand the given expression:

$$\begin{aligned}
\nabla \cdot \left(\underline{\underline{T}} \cdot \boldsymbol{u}\right) - \boldsymbol{u} \cdot \left(\nabla \cdot \underline{\underline{T}}\right) &= (\partial_i \boldsymbol{e}_i) \cdot ((T_{jk} \boldsymbol{e}_j \boldsymbol{e}_k) \cdot (u_m \boldsymbol{e}_m)) - (u_i \boldsymbol{e}_i) \cdot ((\partial_j \boldsymbol{e}_j) \cdot (T_{mk} \boldsymbol{e}_m \boldsymbol{e}_k)) \\
&= (\partial_i \boldsymbol{e}_i) \cdot (T_{jk} u_k \boldsymbol{e}_j) - (u_i \boldsymbol{e}_i) \cdot (\partial_j T_{jk} \boldsymbol{e}_k) \\
&= \partial_i \left(T_{jk} u_k\right) \delta_{ij} - u_i \partial_j T_{jk} \delta_{ik} \\
&= \partial_j \left(T_{jk} u_k\right) - u_k \partial_j T_{jk} \\
&= T_{jk} \partial_j u_k \\
&= \underline{\underline{T}} \colon \nabla \boldsymbol{u}.
\end{aligned}$$

Now, we decompose $\partial_j u_k$ as

$$\partial_j u_k = \frac{1}{2} \left(\partial_j u_k + \partial_k u_j\right) + \frac{1}{2} \left(\partial_j u_k - \partial_k u_j\right) = D_{jk} + \Omega_{jk},$$

where $\underline{\underline{D}} = D_{jk}$ is symmetric and $\underline{\underline{\Omega}} = \Omega_{jk}$ is skew-symmetric; in tensor notation we have

$$\nabla \boldsymbol{u} = \underline{\underline{D}} + \underline{\underline{\Omega}}.$$

The desired result follows from the fact that the double contraction between symmetric and skew-symmetric tensors is zero. Indeed,

$$\underline{\underline{T}} \colon \underline{\underline{\Omega}} = T_{ij} \Omega_{ij} = -T_{ji} \Omega_{ji} = -T_{ij} \Omega_{ij}.$$

■

Note that the kinetic part of the principle of conservation of energy is somewhat hidden in the Cauchy's equation of motion (2.2.9). Let us summarise all five equations that result from the conservation principles:

$$\begin{aligned}
\rho_t + \nabla \cdot (\rho \boldsymbol{u}) &= 0 && \text{(Continuity equation)} \\
\rho \left(\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right) &= \rho \boldsymbol{F}_b + \nabla \cdot \underline{\underline{T}} && \text{(Conservation of linear momentum)} \\
\rho \left(e_t + \boldsymbol{u} \cdot \nabla e\right) &= \underline{\underline{T}} \colon \underline{\underline{D}} - \nabla \cdot \boldsymbol{q}, && \text{(Conservation of internal energy)}
\end{aligned}$$

where $\underline{\underline{D}} = \frac{1}{2} \left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\right)$ and the 14 unknowns are $\rho, \boldsymbol{u}, \underline{\underline{T}}, e, \boldsymbol{q}$.

2.3 Constitutive Laws

To capture the physics neglected from assuming the continuum hypothesis, we need to postulate constitutive equations or laws that relate information on the microscopic scales to the macroscopic scales. These are additional relations among unknowns that model physical processes at the molecular scale, not captured in the continuum assumption. For fluid flows, we postulate constitutive relations between $\underline{\underline{T}}, \boldsymbol{q}$ and the basic quantity of interest $\rho, \boldsymbol{u}, e$.

2.3.1 Stress tensor in a static fluid

We start with the simple case of a static fluid under the influence of gravity, *i.e.* $\boldsymbol{u} \equiv 0$ and $\boldsymbol{F}_b = \boldsymbol{g} = -g\boldsymbol{e}_3$. The Cauchy's equation of motion (2.2.9) reduces to

$$\mathbf{0} = \rho\boldsymbol{g} + \nabla \cdot \underline{\underline{T}}. \tag{2.3.1}$$

If we further assume that the fluid is isothermal, *i.e.* the temperature is uniform, then thermodynamics tells us that the only surface force per unit area is a normal force called pressure p. Consequently,

$$\boldsymbol{n} \cdot \underline{\underline{T}} = -p\boldsymbol{n} = -p\boldsymbol{n} \cdot \underline{\underline{I}} \implies \underline{\underline{T}} = -p\underline{\underline{I}}, \quad p = -\frac{T_{ii}}{3}$$

and (2.3.1) becomes

$$\mathbf{0} = \rho\boldsymbol{g} + \nabla \cdot \left(-p\underline{\underline{I}}\right) = \rho\boldsymbol{g} - \nabla p. \tag{2.3.2}$$

It is clear from (2.3.2) that $p = p(z)$ since $\partial_x p = \partial_y p = 0$ and so we are left with solving z-component of (2.3.2):

$$0 = -\rho g - \frac{dp}{dz} \implies p(z) = -\rho g z + Cp(0) - \rho g z.$$

This is the familiar expression for the hydrostatic pressure p, with $\{z = 0\}$ the air-water interface and it completely determines the stress tensor for static fluid under the influence of gravity.

Remark 2.3.1. The equation $\boldsymbol{n} \cdot \underline{\underline{T}} = -p\boldsymbol{n}$ also means that the stress has the same value for all possible orientations of $\boldsymbol{n}$, *i.e.* the stress is isotropic. This is known as Pascal's law and it is a direct consequence of the fact that a fluid element cannot remain at rest under the presence of a shear stress.

2.3.2 Ideal fluid

For a moving fluid $\boldsymbol{u} \neq \mathbf{0}$, the simplest model for the stress tensor is assuming $\underline{\underline{T}} = -p\underline{\underline{I}}$ and there is no internal friction, *i.e.* $\boldsymbol{q} \equiv \mathbf{0}$. Such a fluid is called an **ideal fluid**, and the 6 components of the stress tensor $\underline{\underline{T}}$ is replaced with a single unknown scalar function p. The system of conservation equations reduces to

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\boldsymbol{u}) = 0 \tag{2.3.3a}$$

$$\rho\left(\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla\boldsymbol{u}\right) = -\nabla p + \rho\boldsymbol{g} \tag{2.3.3b}$$

$$\rho\left(e_t + \boldsymbol{u} \cdot \nabla e\right) = -p\nabla \cdot \boldsymbol{u}. \tag{2.3.3c}$$

which is a system of 5 equations with 6 unknowns $\rho, \boldsymbol{u}, p, e$. It remains to specify a relation between p and ρ, e, *i.e.* $p = p(\rho, e)$ from thermodynamics. Such a relation is called an **equation of state**. Another solution to resolve this underdetermined system is to assume that the flow is incompressible, in which the system (2.3.3) reduces to

$$\frac{D\rho}{Dt} = 0$$

$$\rho\left(\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \rho\boldsymbol{g}$$
$$\frac{De}{Dt} = 0.$$

In the special case where the initial fluid density and internal energy are homogeneous, *i.e.* $\rho(\boldsymbol{x},0) = \rho_0, e(\boldsymbol{x},0) = e_0$, we obtain the **incompressible Euler equations**:

$$\rho_0\left(\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \rho_0\boldsymbol{g} \tag{2.3.4a}$$
$$\nabla\cdot\boldsymbol{u} = 0. \tag{2.3.4b}$$

We may rewrite the right-hand side of the Euler equations by defining the **dynamic pressure** $P(\boldsymbol{x},t)$ satisfying

$$-\nabla p + \rho\boldsymbol{g} = -\nabla(p + \rho g z) = -\nabla P.$$

We note that the dynamic pressure may be introduced only if the density is uniform, the gravitational body force *per unit volume* then being representable as the gradient of a scalar quantity. An important feature of the incompressible Euler equations is that we do not require an equation of state. However, the incompressibility assumption has its limitation, as illustrated in the following example.

Example 2.3.2. Consider the pressure-driven flow in a channel of length L, with P_1 and P_2 the pressure at the left and right end respectively and $P_1 > P_2$. Assuming that there is no variation in the z direction, we look for unidirectional flow of the form $\boldsymbol{u} = (u(x,y,t),0,0)$.

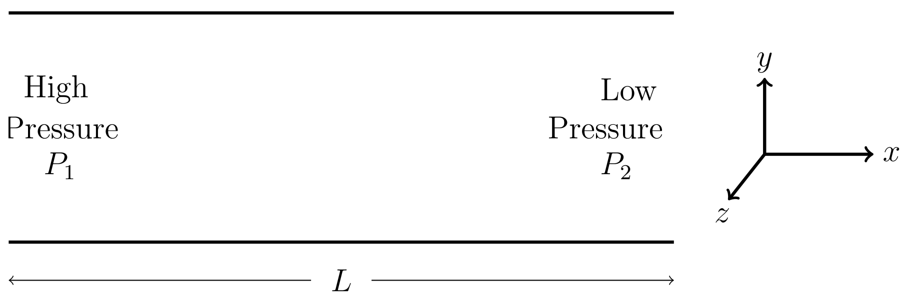

Figure 2.3: Pressure-driven flow in a finite length channel.

The incompressible Euler equations (2.3.4) simplify significantly to

$$\rho u_t = -\partial_x P$$
$$0 = -\partial_y P$$
$$0 = -\partial_z P$$
$$\nabla\cdot\boldsymbol{u} = u_x = 0.$$

It is clear that $P = P(x,t)$ and $u = u(y,t)$. We first solve for the dynamic pressure $P(x,t)$. Differentiating $\rho u_t = -\partial_x P$ with respect to x yields

$$-\partial_{xx}P = \rho(u_t)_x = \rho(u_x)_t = 0$$

$$P(x,t) = a(t)x + b(t).$$

Imposing the boundary conditions $P(0,t) = P_1$ and $P(L,t) = P_2$, we obtain

$$P(x,t) = \left(\frac{P_2 - P_1}{L}\right) x + P_1$$

and the dynamic pressure decreases linearly as expected. Finally we solve for $u(y,t)$.

$$\begin{aligned} \rho u_t &= -\partial_x P = \frac{P_1 - P_2}{L} \\ u_t &= \frac{P_1 - P_2}{\rho L} \\ u(y,t) &= \left(\frac{P_1 - P_2}{\rho L}\right) t + u_0(y). \end{aligned}$$

Assuming $u_0(y)$ is bounded, this model seems physically implausible since $u(y,t) \longrightarrow \infty$ as $t \longrightarrow \infty$. The fundamental reason is that the stress tensor $\underline{\underline{T}} = -p\underline{\underline{I}}$ does not take into account the relative motion between adjacent fluid particles, which can be thought as friction or shear stress between moving layers.

2.3.3 Local decomposition of fluid motion

From the previous example, it seems that we need to understand the microscopic origin of shear stress to capture this missing molecular description. Since there is no shear stress in a static fluid, it is present only if there exists a velocity gradient in the fluid flow. This prompts us to investigate the local velocity variation near any fixed but arbitrary spatial point $\boldsymbol{x}$.

Choose a sufficiently small $\boldsymbol{h} > 0$ and for simplicity assume that the flow is steady. Expanding $\boldsymbol{u}$ around the point $\boldsymbol{x}$ yields

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{h}) &= \boldsymbol{u}(\boldsymbol{x}) + \nabla \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{h} + \mathcal{O}\left(\|h\|^2\right) \\ &= \boldsymbol{u}(\boldsymbol{x}) + \left(\underline{\underline{D}}(\boldsymbol{x}) + \underline{\underline{\Omega}}(\boldsymbol{x})\right) \cdot \boldsymbol{h} + \mathcal{O}\left(\|h\|^2\right), \end{aligned}$$

where

$$\begin{aligned} \underline{\underline{D}}(\boldsymbol{x}) &= \text{ rate of strain/deformation tensor} \\ \underline{\underline{\Omega}}(\boldsymbol{x}) &= \text{ vorticity tensor.} \end{aligned}$$

To see how these names arise, we examine their geometrical meanings in terms of kinematics of the fluid particles. Let $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{h}$. Since $\boldsymbol{x}$ is fixed, we also have

$$\frac{d\boldsymbol{h}}{dt} = \frac{d\boldsymbol{y}}{dt} = \boldsymbol{u}(\boldsymbol{y}) \approx \boldsymbol{u}(\boldsymbol{x}) + \underline{\underline{D}}(\boldsymbol{x}) \cdot \boldsymbol{h} + \underline{\underline{\Omega}}(\boldsymbol{x}) \cdot \boldsymbol{h},$$

which is linear in $\boldsymbol{h}$.

1. If $\partial_t \boldsymbol{h} = \boldsymbol{u}(\boldsymbol{x})$, then $\boldsymbol{h} = \boldsymbol{h}_0 + \boldsymbol{u}(\boldsymbol{x})t$ and this corresponds to a rigid translation.

2. Consider $\partial_t \boldsymbol{h} = \underline{\underline{\Omega}}(\boldsymbol{x}) \cdot \boldsymbol{h}$. Since $\underline{\underline{\Omega}}$ is skew-symmetric, it has only 3 components which are the components of the curl vector $\nabla \times \boldsymbol{u}$. Define the **vorticity** $\boldsymbol{\omega}(\boldsymbol{x})$ as

$$\boldsymbol{\omega}(\boldsymbol{x}) = \nabla \times \boldsymbol{u} = \begin{bmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}.$$

Consequently,

$$\underline{\underline{\Omega}} = \frac{1}{2} \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

and

$$\frac{d\boldsymbol{h}}{dt} = \underline{\underline{\Omega}}(\boldsymbol{x}) \cdot \boldsymbol{h} = \frac{1}{2}\Big(\boldsymbol{\omega}(\boldsymbol{x}) \times \boldsymbol{h}\Big).$$

The geometrical interpretation of the cross product tells us that the vector $\partial_t \boldsymbol{h}$ is rotating about the axis $\omega/\|\omega\|$ with angular velocity $\|\omega\|/2$. Moreover, it is a rigid rotation since the length of $\boldsymbol{h}$ is constant:

$$\frac{d}{dt}\|\boldsymbol{h}\|^2 = \frac{d}{dt}(\boldsymbol{h} \cdot \boldsymbol{h}) = 2\boldsymbol{h} \cdot \frac{d\boldsymbol{h}}{dt} = \boldsymbol{h} \cdot (\boldsymbol{\omega} \times \boldsymbol{h}) = 0.$$

The vorticity $\boldsymbol{\omega}$ acts as a measure of the local spinning motion.

3. Suppose $\partial_t \boldsymbol{h} = \underline{\underline{D}}(\boldsymbol{x}) \cdot \boldsymbol{h}$. Since $\underline{\underline{D}}(\boldsymbol{x})$ is symmetric, there exists orthonormal eigenvectors $\{\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3\}$ with corresponding eigenvalues d_1, d_2, d_3 such that

$$\underline{\underline{D}}(\boldsymbol{x})\tilde{\boldsymbol{e}}_j = d_j \tilde{\boldsymbol{e}}_j, \quad j = 1, 2, 3.$$

Since the eigenvectors form a basis of $\mathbb{R}^3$, we can decompose $\boldsymbol{h}(t)$ as

$$\boldsymbol{h}(t) = \tilde{h}_1(t)\tilde{\boldsymbol{e}}_1 + \tilde{h}_2(t)\tilde{\boldsymbol{e}}_2 + \tilde{h}_3(t)\tilde{\boldsymbol{e}}_3.$$

It follows from linearity of $\underline{\underline{D}}(\boldsymbol{x})$ that

$$\sum_{j=1}^{3} \frac{d\tilde{h}_j(t)}{dt}\tilde{\boldsymbol{e}}_j = \underline{\underline{D}}(\boldsymbol{x}) \left(\sum_{j=1}^{3} \tilde{h}_j(t)\tilde{\boldsymbol{e}}_j \right) = \sum_{j=1}^{3} \Big(d_j \tilde{h}_j(t) \Big) \tilde{\boldsymbol{e}}_j,$$

and using linear independency of $\{\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3\}$ we obtain a system of 3×3 uncoupled ODEs

$$\frac{d\tilde{h}_j(t)}{dt} = d_j \tilde{h}_j(t), \quad j = 1, 2, 3.$$

This says that a fluid element is either stretched or shrinked in directions $\tilde{\boldsymbol{e}}_j$ according to the sign of d_j and deforms into a parallelepiped. Because of this, $\underline{\underline{D}}(\boldsymbol{x})$ is called the rate of strain/deformation tensor and the eigenvectors $\{\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3\}$ are called the **principal direction** of the rate of deformation tensor.

Hence, the flow in the neighbourhood of any point $\boldsymbol{x}$ is decomposed into a rigid translation $\boldsymbol{u}(\boldsymbol{x})$, a rigid spinning motion $\underline{\underline{\Omega}}(\boldsymbol{x}) \cdot \boldsymbol{h}$ and a flow involving deformation $\underline{\underline{D}}(\boldsymbol{x}) \cdot \boldsymbol{h}$. Observe that

$$\begin{aligned}\frac{d}{dt}\left(\tilde{h}_1\tilde{h}_2\tilde{h}_3\right) &= \frac{d\tilde{h}_1}{dt}\tilde{h}_2\tilde{h}_3 + \tilde{h}_1\frac{d\tilde{h}_2}{dt}\tilde{h}_3 + \tilde{h}_1\tilde{h}_2\frac{d\tilde{h}_3}{dt} \\ &= (d_1 + d_2 + d_3)\left(\tilde{h}_1\tilde{h}_2\tilde{h}_3\right) \\ &= \operatorname{tr}\left(\underline{\underline{D}}\right)\left(\tilde{h}_1\tilde{h}_2\tilde{h}_3\right) \\ &= (\nabla \cdot \boldsymbol{u})\left(\tilde{h}_1\tilde{h}_2\tilde{h}_3\right).\end{aligned}$$

Since $\underline{\underline{D}}(\boldsymbol{x})$ is unitary equivalent to the diagonal matrix $\Lambda = \operatorname{diag}(d_1, d_2, d_3)$, we obtain

$$d_1 + d_2 + d_3 = \operatorname{tr}(\Lambda) = \operatorname{tr}\left(\underline{\underline{D}}\right) = D_{11} + D_{22} + D_{33} = \nabla \cdot \boldsymbol{u}.$$

Consequently, the relative rate of change of volume of fluid element is

$$\frac{1}{\tilde{h}_1\tilde{h}_2\tilde{h}_3}\frac{d}{dt}\left(\tilde{h}_1\tilde{h}_2\tilde{h}_3\right) = \nabla \cdot \boldsymbol{u}.$$

If the flow is incompressible, then

$$\frac{d}{dt}\left(\tilde{h}_1\tilde{h}_2\tilde{h}_3\right) = 0,$$

which means the fluid element stretches at some direction and shrinks at the other direction.

2.3.4 Stokes assumption for Newtonian fluid

In general, we may write the stress tensor $\underline{\underline{T}}$ as

$$\underline{\underline{T}} = -p\underline{\underline{I}} + \underline{\underline{\sigma}}.$$

We expect that $\underline{\underline{\sigma}}$ depends only on its symmetric part since both $\underline{\underline{T}}$ and $\underline{\underline{I}}$ are symmetric. It is also clear that $\underline{\underline{\sigma}}$ is due to the fluid motion because we must recover $\underline{\underline{T}} = -p\underline{\underline{I}}$ in the case of static fluid. To obtain a constitutive relation for the stress of fluids that depends on relative motion, Sir George Stokes (1845) postulates the following:

1. $\underline{\underline{\sigma}}$ should vanish if the flow involves no deformation of fluid elements;
2. $\underline{\underline{T}}$ does not depend explicitly on the location $\boldsymbol{x}$ and time t (Homogeneous);
3. the relationship between $\underline{\underline{\sigma}}$ and the velocity gradient should be isotropic, as the physical properties of the fluid are assumed to show no preferred direction (Isotropic);
4. $\underline{\underline{T}}$ is a continuous function of $\underline{\underline{D}}$ and is otherwise independent of the fluid motion (Local);
5. $\underline{\underline{T}}$ depends linearly on $\underline{\underline{D}}$ (Linear).

These suggest that $\underline{\underline{T}}$ has the form

$$T_{ij} = -p\delta_{ij} + C_{ijpq}D_{pq},$$

where C_{ijpq} are 81 constants.

2.3.5 Cartesian tensors

A major requirement for an object to be a tensor is that it transforms in a specific way under changes of Cartesian coordinates. This is motivated by the fact that any physical laws expressed in any two different Cartesian coordinate systems (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) must be consistent. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and $\{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3\}$ be unit basis vectors in the unprimed and primed system respectively. Consider a point $\boldsymbol{P}$ in $\mathbb{R}^3$, with coordinates (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) in the unprimed and primed system respectively. Then

$$\boldsymbol{P} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 = x'_1\boldsymbol{e}'_1 + x'_2\boldsymbol{e}'_2 + x'_3\boldsymbol{e}'_3.$$

How are $\{x_1, x_2, x_3\}$ and $\{x'_1, x'_2, x'_3\}$ related? Taking the inner product of $\boldsymbol{P}$ against $\boldsymbol{e}_j$ yields

$$x_i = \boldsymbol{e}_i \cdot \boldsymbol{P} = \boldsymbol{e}_i \cdot \left(x'_j\boldsymbol{e}'_j\right) = \left(\boldsymbol{e}_i \cdot \boldsymbol{e}'_j\right) x'_j = \ell_{ij}x'_j = \sum_{j=1}^{3} \ell_{ij}x'_j,$$

where ℓ_{ij} is cosine of the angle between unit vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}'_j$. Similarly, we have

$$x'_i = \boldsymbol{e}'_i \cdot \boldsymbol{P} = \boldsymbol{e}'_i \cdot (x_j\boldsymbol{e}_j) = (\boldsymbol{e}'_i \cdot \boldsymbol{e}_j)\, x_j = \ell_{ji}x_j = \sum_{j=1}^{3} \ell_{ji}x_j.$$

It can be shown that

$$\ell_{ij}\ell_{kj} = \ell_{ji}\ell_{jk} = \delta_{ik} \implies \underline{\underline{L}}\,\underline{\underline{L}}^T = \underline{\underline{L}}^T\,\underline{\underline{L}} = \underline{\underline{I}},$$

where $\underline{\underline{L}}$ is the second-order tensor with components ℓ_{ij}.

Definition 2.3.3. An nth-order tensor C in $\mathbb{R}^3$, $n = 1, 2, 3, \ldots$ is an object such that

1. In any Cartesian coordinate system, there is a rule that associates C with a unique ordered set of 3^n scalars $C_{i_1 i_2 \ldots i_n}$, called components of C in that coordinate system.
2. If $C_{i_1 i_2 \ldots i_n}$ and $C_{q_1 q_2 \ldots q_n}$ are the components of C with respect to two different Cartesian coordinate systems, then

$$C_{i_1 i_2 \ldots i_n} = l_{i_1 q_1} l_{i_2 q_2} \ldots\ldots l_{i_{n-1} q_{n-1}} l_{i_n q_n} C_{q_1 q_2 \ldots q_n}.$$

It follows from the definition that any second-order tensor $\underline{\underline{T}}$ must satisfy the consistency rule

$$T_{ij} = l_{ip} l_{jq} T'_{pq}.$$

To see this, consider the surface traction vector $\boldsymbol{t} = \boldsymbol{n} \cdot \underline{\underline{T}}$. In the unprimed and primed systems, we have

$$t_i = T_{ij}n_j = \sum_{j=1}^{3} T_{ij}n_j, \quad t_i = \ell_{ip}t'_p = \sum_{p=1}^{3} \ell_{ip}t'_p$$

$$t'_p = T'_{pq}n'_q = \sum_{q=1}^{3} T'_{pq}n'_q, \quad n_j = \ell_{jq}n'_q = \sum_{q=1}^{3} \ell_{jq}n'_q.$$

If these are consistent, we must have

$$\begin{aligned} t_i = \ell_{ip} t'_p &= \ell_{ip} T'_{pq} n'_q \\ &= \ell_{ip} T'_{pq} \ell_{jq} n_j \\ &= T_{ij} n_j \\ \implies (T_{ij} - \ell_{ip}\ell_{jq} T'_{pq}) n_j &= 0. \end{aligned}$$

For this to be hold for n_j, we must then have $T_{ij} = l_{ip} l_{jq} T'_{pq}$ as expected.

2.3.6 Stress tensor for Newtonian fluid

Going back to the stress tensor

$$T_{ij} = -p\delta_{ij} + C_{ijpq} D_{pq},$$

we argue that C_{ijpq} must be a fourth-order tensor since T_{ij} and δ_{ij} are both second-order tensors. In fact, it must be isotropic from Stokes' isotropic assumption. Now, any fourth-order isotropic tensor can be written as

$$\begin{aligned} C_{ijpq} &= a\delta_{ij}\delta_{pq} + b\delta_{ip}\delta_{jq} + c\delta_{iq}\delta_{jp} \\ &= \lambda\delta_{ij}\delta_{pq} + \mu\Big[\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}\Big] + \kappa\Big[\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}\Big]. \end{aligned}$$

Since $\underline{\underline{T}}$ and $\underline{\underline{I}}$ are both symmetric, we must have

$$C_{ijpq} D_{pq} = C_{jipq} D_{pq} \implies C_{ijpq} = C_{jipq} \quad \text{for all } i, j, p, q = 1, 2, 3.$$

Therefore $\kappa = 0$ and

$$\begin{aligned} T_{ij} &= -p\delta_{ij} + \lambda\delta_{ij}\delta_{pq} D_{pq} + \mu\Big(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}\Big) D_{pq} \\ &= -p\delta_{ij} + \lambda\delta_{ij} \operatorname{tr}(\underline{\underline{D}}) + \mu\Big(D_{ij} + D_{ji}\Big) \\ &= \Big(-p + \lambda\nabla\cdot\boldsymbol{u}\Big)\delta_{ij} + 2\mu D_{ij}. \end{aligned}$$

The parameter μ is the **shear viscosity** and the parameter λ relates to the bulk viscosity which is important only when the fluid is being rapidly compressed or expanded.

Definition 2.3.4. A **Newtonian fluid** is a fluid with stress tensor of the form

$$\underline{\underline{T}} = \Big(-p + \lambda\nabla\cdot\boldsymbol{u}\Big)\underline{\underline{I}} + 2\mu\underline{\underline{D}}, \tag{2.3.5}$$

for some scalars λ, μ.

Remark 2.3.5. Because the trace of any second-order tensors is invariant with respect to a change of basis, we can define the "mechanical" pressure P as

$$P_m = -\frac{1}{3}\operatorname{tr}(\underline{\underline{T}}) = \frac{1}{3}\Big[T_{11} + T_{22} + T_{33}\Big] \tag{2.3.6}$$

and it is related to the thermodynamic equilibrium pressure p according to

$$P_m = p - \frac{1}{3}\Big(3\lambda + 2\mu\Big)\nabla \cdot \boldsymbol{u} = p - \eta\nabla \cdot \boldsymbol{u}, \tag{2.3.7}$$

where

$$\eta = \lambda + \frac{2}{3}\mu = \frac{p - P_m}{\nabla \cdot \boldsymbol{u}} \tag{2.3.8}$$

is the **bulk viscosity**. Consequently, the Newtonian stress tensor can also be written as

$$\underline{\underline{T}} = -p\underline{\underline{I}} + 2\mu\left(\underline{\underline{D}} - \frac{1}{3}(\nabla \cdot \boldsymbol{u})\underline{\underline{I}}\right) + \eta(\nabla \cdot \boldsymbol{u})\underline{\underline{I}}, \tag{2.3.9}$$

or, in terms of P_m,

$$\underline{\underline{T}} = -P_m\underline{\underline{I}} + 2\mu\left(\underline{\underline{D}} - \frac{1}{3}(\nabla \cdot \boldsymbol{u})\underline{\underline{I}}\right). \tag{2.3.10}$$

Writing the stress tensor as $\underline{\underline{T}} = -P_m\underline{\underline{I}} + \underline{\underline{\tau}}$, these two stress tensors are known as the *mean normal stress tensor* and the *deviatoric stress tensor* respectively.

2.4 Isothermal, Incompressible Navier-Stokes Equations

Recall the three conservation equations

$$\begin{aligned} \partial_t\rho + \nabla \cdot (\rho\boldsymbol{u}) &= 0 \\ \rho(\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla\boldsymbol{u}) &= \rho\boldsymbol{F}_b + \nabla \cdot \underline{\underline{T}} \\ \rho(e_t + \boldsymbol{u} \cdot \nabla e) &= \underline{\underline{T}} \colon \underline{\underline{D}} - \nabla \cdot \boldsymbol{q} \end{aligned}$$

They form a system of 5 equations with 14 unknowns $\rho, \boldsymbol{u}, e, \boldsymbol{q}$ and six components of $\underline{\underline{T}}$. The three constitutive equations are

$$\begin{aligned} \underline{\underline{T}} &= \Big(-p + \lambda\nabla \cdot \boldsymbol{u}\Big)\underline{\underline{I}} + 2\mu\underline{\underline{D}} && \text{(Newtonian fluid: 6 equations, 1 unknown } p) \\ \boldsymbol{q} &= -k\nabla T && \text{(Fourier law: 3 equations, 1 unknown } T) \\ p &= p(\rho, e) && \text{(Equation of state: 1 equation)} \end{aligned}$$

Suppose the initial fluid density is homogeneous. Under isothermal conditions, the temperature T is constant and so $\boldsymbol{q} = \boldsymbol{0}$. The conservation of energy equation reduces to

$$\rho(e_t + \boldsymbol{u} \cdot \nabla e) = \underline{\underline{T}} \colon \underline{\underline{D}},$$

which says that e can be determined by knowing $\underline{\underline{T}}$ and $\underline{\underline{D}}$, or equivalently, the pressure p and the velocity field $\boldsymbol{u}$. Together with the incompressibility condition and $\boldsymbol{F}_b = \boldsymbol{g} = -g\boldsymbol{e}_3$, we arrive at the **isothermal, incompressible Navier-Stokes equations**

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla\boldsymbol{u}\right) = -\nabla P + \mu\Delta\boldsymbol{u} \tag{2.4.1a}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.4.1b}$$

where P is the *dynamic pressure* and μ is the **dynamic viscosity**. We mention that the pressure p is the Lagrange multiplier of the linear constraint $\nabla \cdot \boldsymbol{u} = 0$, and this is nontrivial to prove. (A proof of this statement in the case of the steady Stokes equation can be found in [Oza17].) For nonuniform density, we replace the incompressibility condition with the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho = 0$$

and an equation of state for the pressure p is needed since (2.4.1) now has 4 equations with 5 unknowns. To see how the vector Laplacian term arises in (2.4.1), note that

$$\begin{aligned} 2\nabla \cdot \underline{\underline{D}} = 2\left(\partial_i \boldsymbol{e}_i\right) \cdot \left(D_{jk} \boldsymbol{e}_j \boldsymbol{e}_k\right) &= 2\partial_i D_{jk} \delta_{ij} \boldsymbol{e}_k \\ &= 2\partial_j D_{jk} \boldsymbol{e}_k \\ &= \partial_j \left(\partial_j u_k + \partial_k u_j\right) \boldsymbol{e}_k. \end{aligned}$$

For any $k \in \{1, 2, 3\}$,

$$\begin{aligned} \partial_j \left(\partial_j u_k + \partial_k u_j\right) \boldsymbol{e}_k &= \left(\partial_j \partial_j u_k + \partial_j \partial_k u_j\right) \boldsymbol{e}_k \\ &= \left(\Delta u_k + \partial_k \partial_j u_j\right) \boldsymbol{e}_k \\ &= \left[\Delta u_k + \partial_k \left(\nabla \cdot \boldsymbol{u}\right)\right] \boldsymbol{e}_k, \end{aligned}$$

where we assume that $\boldsymbol{u}$ is C^2 so that we can interchange the order of partial derivatives. The incompressibility condition then gives

$$\partial_j \left(\partial_j u_k + \partial_k u_j\right) \boldsymbol{e}_k = \left(\Delta u_k\right) \boldsymbol{e}_k \implies 2\mu \nabla \cdot \underline{\underline{D}} = \mu \Delta \boldsymbol{u}.$$

Remark 2.4.1. If the inertial effects are negligible, then we may eliminate the term $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ and obtain the (unsteady) Stokes equations. If the viscous effects are negligible instead, then we may eliminate the Laplacian term $\mu \Delta \boldsymbol{u}$ and recover the incompressible Euler's equations; the fluid is said to be **inviscid** in this case.

2.5 Boundary Conditions

To solve the incompressible Navier-Stokes equations, we must specify initial and boundary conditions. For an infinite domain, we impose the *far-field condition*: $\boldsymbol{u} \longrightarrow 0$ as $\boldsymbol{x} \longrightarrow \infty$. For a bounded domain, we distinguish two types of boundary conditions (BCs):

1. Velocity BCs.

   (a) <u>Kinematic BC:</u> It is clear that the component of the velocity normal to the boundary S must be continuous across S. This follows from the conservation of mass, since the boundary will accumulate mass otherwise. For a stationary solid wall, this is known as the **no-penetration** or **no-flux** boundary condition:

   $$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on the wall.}$$

   For a moving solid wall,

   $$\boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{u}_{\text{wall}} \cdot \boldsymbol{n} \quad \text{on the wall.}$$

For a moving (deformable) boundary, the kinematic BC can be rephrased in the following equivalent way. Define the time-dependent boundary as the level set of some implicit function $F(\boldsymbol{x}, t) = 0$. Fluid particles on the surface must remain on the surface as time evolves and consequently the material derivative of F must vanish, *i.e.*

$$\frac{DF}{Dt} = 0 \quad \text{on the moving boundary.}$$

This is applicable to water waves problem where we have a moving interface between water and air, called the free surface.

(b) Tangential BC: This is only applicable to viscous fluid but not ideal fluid, essentially because ideal fluid is "slippery". Suppose that the pressure varies near the boundary along the wall. The only force a fluid element can experience is a pressure force associated with the pressure gradient. If such gradient at the wall is tangent to the wall, then fluid will be accelerated and there must be a tangential velocity at the wall. This suggests that we cannot place any restriction on the tangential velocity at a solid wall. In terms of the well-posedness of PDEs, we require another BC in addition to the no-penetration BC due to the Laplacian term that only appears for viscous fluid.

i. We demand that the component of the velocity tangential to the boundary S must also be continuous. This is known as the **no-slip** condition and it cannot be derived from fundamental physics. For a moving solid wall, it takes the form

$$\boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n} = \boldsymbol{u}_{\text{wall}} - (\boldsymbol{u}_{\text{wall}} \cdot \boldsymbol{n})\boldsymbol{n} \quad \text{on the wall.}$$

Note that this breaks down on the molecular level but it remains a good approximation for small-scale problems.

ii. The no-slip BC causes a problem in the moving contact-line problem, since it leads to a infinite-force singularity at the moving contact line. This can be resolved by introducing the **Navier-slip** condition:

$$\boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n} = \beta\Big[\underline{\underline{T}} \cdot \boldsymbol{n} - \Big[(\underline{\underline{T}} \cdot \boldsymbol{n}) \cdot \boldsymbol{n}\Big]\boldsymbol{n}\Big] \quad \text{on the wall,}$$

where β is the (emperical) slip coefficient. This says that the tangential (slip) velocity is proportional to the tangential component of the stress on the wall. For a shear flow $(u(y), 0, 0)$ of a Newtonian fluid, this slip condition translates to $u = \beta\mu u_y$, where μu_y is the shear stress on the wall.

2. Stress BCs. This is considered on free surface problems where it involves a fluid-fluid boundary S. The tangential stress across S should be continuous, and the condition on normal stress across S is somewhat more complicated because of the presence of surface tension.

2.6 The Reynolds Number

To compare the magnitude of terms in the incompressible Navier-Stokes equations with actual numbers, we non-dimensionalise the equations. To this end, let L, U, t_c, P_c be characteristic length, velocity, time, pressure scales respectively and define the dimensionless variables

$\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}}, \tilde{t}, \tilde{P}$ as follows:

$$\tilde{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L}, \quad \tilde{\boldsymbol{u}} = \frac{\boldsymbol{u}}{U}, \quad \tilde{t} = \frac{t}{t_c}, \quad \tilde{P} = \frac{P}{P_c}.$$

From chain rule we obtain

$$\frac{\partial}{\partial t} = \frac{1}{t_c}\frac{\partial}{\partial \tilde{t}}, \quad \frac{\partial}{\partial \boldsymbol{x}} = \frac{1}{L}\frac{\partial}{\partial \tilde{\boldsymbol{x}}}.$$

Denote the dimensionless differential operator $\tilde{\nabla} = \partial_{\tilde{x}_i}\boldsymbol{e}_i$. Component-wise, we have

$$\begin{aligned}
(\boldsymbol{u}_t)_i = \frac{\partial u_i}{\partial t} &= \frac{U}{t_c}\frac{\partial \tilde{u}_i}{\partial \tilde{t}} = \frac{U}{t_c}(\tilde{\boldsymbol{u}}_{\tilde{t}})_i \\
(\boldsymbol{u}\cdot\nabla\boldsymbol{u})_i = \boldsymbol{u}\cdot\nabla u_i &= \sum_{j=1}^{3} u_j\partial_{x_j}u_i \\
&= U^2\sum_{j=1}^{3}\tilde{u}_j\partial_{x_j}\tilde{u}_i \\
&= \frac{U^2}{L}\sum_{j=1}^{3}\tilde{u}_j\partial_{\tilde{x}_j}\tilde{u}_i \\
&= \frac{U^2}{L}\left(\tilde{\boldsymbol{u}}\cdot\tilde{\nabla}\tilde{\boldsymbol{u}}\right)_i \\
(\nabla P)_i = \frac{\partial P}{\partial x_i} &= \frac{P_c}{L}\frac{\partial \tilde{P}}{\partial x_i} = \frac{P_c}{L}\left(\tilde{\nabla}\tilde{P}\right)_i \\
(\Delta\boldsymbol{u})_i = \Delta u_i &= \sum_{j=1}^{3}\partial_{x_j}^2 u_i \\
&= U\sum_{j=1}^{3}\partial_{x_j}^2\tilde{u}_i \\
&= \frac{U}{L^2}\sum_{j=1}^{3}\partial_{\tilde{x}_j}^2\tilde{u}_i \\
&= \frac{U}{L^2}\left(\tilde{\Delta}\tilde{\boldsymbol{u}}\right)_i .
\end{aligned}$$

The dimensionless incompressible Navier-Stokes equations are

$$\rho\left[\left(\frac{U}{t_c}\right)\frac{\partial\tilde{\boldsymbol{u}}}{\partial\tilde{t}} + \underbrace{\left(\frac{U^2}{L}\right)\left(\tilde{\boldsymbol{u}}\cdot\tilde{\nabla}\tilde{\boldsymbol{u}}\right)}_{\text{inertial term}}\right] = -\left(\frac{P_c}{L}\right)\tilde{\nabla}\tilde{P} + \underbrace{\left(\frac{\mu U}{L^2}\right)\tilde{\Delta}\tilde{\boldsymbol{u}}}_{\text{viscous term}}$$

$$\frac{U}{L}\tilde{\nabla}\cdot\tilde{\boldsymbol{u}} = 0.$$

Comparing the magnitudes of inertial and viscous terms, we obtain

$$\frac{|\text{inertial term}|}{|\text{viscous term}|} = \mathcal{O}\left(\frac{\rho U^2/L}{\mu U/L^2}\right) = \mathcal{O}\left(\frac{\rho U L}{\mu}\right) = \mathcal{O}(\text{Re}),$$

where the dimensionless number Re is the **Reynolds number**, having the form

$$\mathrm{Re} = \frac{\rho U L}{\mu} = \frac{UL}{\nu}.$$

Thus, the Reynolds number prescribes the relative magnitudes of inertial and viscous forces in the system.

Multiplying the dimensioness Cauchy momentum equation with $L/\rho U^2$, we get

$$\left(\frac{L}{t_c U}\right)\frac{\partial \tilde{\boldsymbol{u}}}{\partial \tilde{t}} + \tilde{\boldsymbol{u}} \cdot \tilde{\nabla}\tilde{\boldsymbol{u}} = -\left(\frac{P_c}{LU^2}\right)\tilde{\nabla}\tilde{P} + \frac{1}{\mathrm{Re}}\tilde{\Delta}\tilde{\boldsymbol{u}}.$$

Define $t_c = U/L$ and $P_c = \rho U^2$, this reduces to

$$\frac{\partial \tilde{\boldsymbol{u}}}{\partial \tilde{t}} + \tilde{\boldsymbol{u}} \cdot \tilde{\nabla}\tilde{\boldsymbol{u}} = -\tilde{\nabla}\tilde{P} + \frac{1}{\mathrm{Re}}\tilde{\Delta}\boldsymbol{u}.$$

There are two limiting cases:

1. The case $\mathrm{Re} \gg 1$ suggests that the viscous effects are negligible and there is a balance between the inertial and pressure terms. We may approximate the dimensionless incompressible Navier-Stokes equations with the dimensionless Euler equations:

$$\tilde{\boldsymbol{u}}_{\tilde{t}} + \tilde{\boldsymbol{u}} \cdot \tilde{\nabla}\tilde{\boldsymbol{u}} = -\tilde{\nabla}\tilde{P}.$$

   However, viscous effects become important in thin boundary layers over which the flow velocity undergoes a smooth but rapid adjustment to precisely zero - corresponding to no slip. The velocity gradients in this thin boundary layer are so large that the viscous stress becomes significant even though μ is small enough for viscous effects to be negligible elsewhere in the flow.

2. For $\mathrm{Re} \ll 1$, we have

$$\mathrm{Re}\left[\frac{\partial \tilde{\boldsymbol{u}}}{\partial \tilde{t}} + \tilde{\boldsymbol{u}} \cdot \tilde{\nabla}\tilde{\boldsymbol{u}}\right] = -\tilde{\nabla}\tilde{P} + \tilde{\Delta}\tilde{\boldsymbol{u}},$$

   where the characteristic pressure scale is now taken to be $P_c = \rho U^2/\mathrm{Re}$. This suggests that the inertial effects are negligible and leads to the so-called **Stokes flow**:

$$\begin{aligned} -\tilde{\nabla}\tilde{P} + \tilde{\Delta}\tilde{\boldsymbol{u}} &= 0 \\ \tilde{\nabla} \cdot \tilde{\boldsymbol{u}} &= 0. \end{aligned}$$

   We cannot scaled away the pressure term because we need it to satisfy the incompressiblity condition.

2.7 Bernoulli's Theorem

Take the body force to be conservative, that is, $\boldsymbol{F}_b = -\nabla\chi$ for some scalar potential χ. Using the vector identity

$$\boldsymbol{u} \cdot \nabla\boldsymbol{u} = (\nabla \times \boldsymbol{u}) \times \boldsymbol{u} + \frac{1}{2}\nabla|\boldsymbol{u}|^2$$

$$= \boldsymbol{\omega} \times \boldsymbol{u} + \frac{1}{2}\nabla|\boldsymbol{u}|^2$$

we may rewrite the Cauchy momentum equation in (2.4.1) as

$$\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{u} &= -\frac{1}{\rho}\nabla p - \nabla\chi - \frac{1}{2}\nabla|\boldsymbol{u}|^2 - \frac{\nu}{\rho}\Delta\boldsymbol{u} \\
\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{u} &= -\nabla\left(\frac{p}{\rho} + \chi + \frac{1}{2}|\boldsymbol{u}|^2\right) - \frac{\nu}{\rho}\Delta\boldsymbol{u} \\
\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{u} &= -\nabla H + \nu\Delta\boldsymbol{u},
\end{aligned} \tag{2.7.1}$$

where $\nu = \mu/\rho$ is the **kinematic viscosity**. If the flow is steady, then (2.7.1) reduces to

$$\boldsymbol{\omega} \times \boldsymbol{u} = -\nabla H + \nu\Delta\boldsymbol{u}. \tag{2.7.2}$$

Taking the dot product of (2.7.2) against $\boldsymbol{u}$ gives

$$\boldsymbol{u} \cdot \nabla H = -\boldsymbol{u} \cdot \Big((\boldsymbol{\omega} \times \boldsymbol{u}) + \nu\Delta\boldsymbol{u} \Big) = \nu\boldsymbol{u} \cdot \Delta\boldsymbol{u}. \tag{2.7.3}$$

Consequently, H decreases in the flow direction when $\boldsymbol{u}\cdot\Delta\boldsymbol{u} < 0$, *i.e.* when the local net viscous force per unit volume tends to decelerate the fluid and work is done against viscous forces as an element of fluid moves along a streamtube; similarly H increases along the streamline when the net viscous force tends to accelerate the fluid [Bat00, Section 5.1].

In the case of an ideal fluid, (2.7.3) reduces to $\boldsymbol{u} \cdot \nabla H = 0$ which means the directional derivative of H along the flow field $\boldsymbol{u}$ is zero. This is known as **Bernoulli streamline theorem**:

For a steady flow of an ideal fluid, subject to a conservative force $\boldsymbol{F}_b = -\nabla\chi$, *the function* $H = \dfrac{p}{\rho} + \chi + \dfrac{1}{2}|\boldsymbol{u}|^2$ *is constant along a streamline.*

In particular, an increase of the fluid velocity occurs simultaneously with a decrease in the (static) pressure or a decrease in the fluid's potential energy, along a given streamline. Here, $p, \chi, \boldsymbol{u}$ depends on the particular point on the chosen streamline but the constant depends only on that streamline. We point out that the theorem says nothing more than H being constant along a given streamline, so H may have different constants on different streamlines. However, H is constant everywhere for **irrotational** flow, *i.e.* when the flow field $\boldsymbol{u}$ has zero curl $\boldsymbol{\omega} \equiv 0$.

For a steady irrotational flow of an ideal fluid, subject to a conservative force $\boldsymbol{F}_b = -\nabla\chi$, *the function* $H = \dfrac{p}{\rho} + \chi + \dfrac{1}{2}|\boldsymbol{u}|^2$ *is constant everywhere.*

2.8 Vorticity Equation

As it turns out, the net viscous force on an element of incompressible fluid is determined by the local gradients of vorticity since

$$\nabla \times \boldsymbol{\omega} = \nabla \times (\nabla \times \boldsymbol{u}) = \nabla(\nabla \cdot \boldsymbol{u}) - \Delta\boldsymbol{u} = -\Delta\boldsymbol{u}. \tag{2.8.1}$$

This is rather surprising since we know from the Newtonian stress tensor that the viscous stress is generated solely by deformation of the fluid and is independent of the local vorticity, but the explanation is wholly a matter of kinematics [Bat00, Section 3.3]. The rate of deformation tensor $\underline{\underline{D}}$ and the vorticity $\boldsymbol{\omega}$ play independent roles in the generation of stress, but certain spatial derivatives of $\underline{\underline{D}}$ are identically related to certain derivatives of $\boldsymbol{\omega}$ through the vector identity used in (2.8.1). It follows from (2.8.1) that the viscous distribution of vorticity is pivotal in understanding the evolution of large Reynolds number flow , *i.e.* when the fluid viscosity is sufficiently small.

The **vorticity transport equation** is obtained as follows. Taking the curl of the Cauchy momentum equation in the form of (2.7.1) and using the fact that $\nabla \times (\nabla f) = 0$ for any C^2 scalar function f, we find that

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) = \nu \left(\nabla \times \Delta \boldsymbol{u}\right).$$

We use the incompressibility condition and the fact that $\nabla \cdot \boldsymbol{\omega} = 0$ to cancel out terms. Applying the vector identity from (2.8.1) to $\boldsymbol{\omega}$ reduces the viscous term into

$$\nabla \times \Delta \boldsymbol{u} = -\nabla \times (\nabla \times \boldsymbol{\omega}) = -\nabla \left(\nabla \cdot \boldsymbol{\omega}\right) + \Delta \boldsymbol{\omega} = \Delta \boldsymbol{\omega}.$$

For the convection term,

$$\begin{aligned}\nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) &= \boldsymbol{\omega} \nabla \cdot \boldsymbol{u} - \boldsymbol{u} \nabla \cdot \boldsymbol{\omega} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} \\ &= \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}.\end{aligned}$$

Combining all the computations leads to the vorticity equation

$$\begin{aligned}\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} &= \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} + \nu \Delta \boldsymbol{\omega} \\ \frac{D\boldsymbol{\omega}}{Dt} &= \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} + \nu \Delta \boldsymbol{\omega} \qquad (2.8.2)\end{aligned}$$

The pressure term has been eliminated in the vorticity equation, with the price that the vorticity equation contains both $\boldsymbol{u}$ and $\boldsymbol{\omega}$.

1. The term $\dfrac{D\boldsymbol{\omega}}{Dt}$ is the familiar material derivative of $\boldsymbol{\omega}$, describing the rate of change of $\boldsymbol{\omega}$ due to the convection of fluid.

2. The term $\nu \Delta \boldsymbol{\omega}$ accounts for the diffusion of vorticity due to the viscous effects.

3. The term $\boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$ represents the stretching or tilting of vortex tubes due to the folow velocity gradients.

4. The term $\boldsymbol{\omega} \left(\nabla \cdot \boldsymbol{u}\right)$ (which is absent due to the incompressibility condition) describes vortex stretching due to flow compressibility.

In the case of a two-dimensional incompressible inviscid flow, *i.e.* $\boldsymbol{u} = (u(x, y, t), v(x, y, t), 0)$, the vorticity only has one non-zero component

$$\boldsymbol{\omega} = (0, 0, \partial_x v - \partial_y u) = \omega(x, y, t) \boldsymbol{e}_3$$

and

$$\boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = \omega \frac{\partial \boldsymbol{u}}{\partial z} = \mathbf{0}.$$

Consequently, $\dfrac{D\omega}{Dt} = 0$ and so the vorticity ω of each individual fluid particle does not change in time, *i.e.* if $\omega = 0$ at some time t_0, then $\omega \equiv 0$ for all time $t \geq t_0$. This also occurs for a unidirectional flow, say $\boldsymbol{u} = u(y, z, t)\boldsymbol{e}_1$, since we then have

$$\boldsymbol{\omega} = (0, \partial_z u, -\partial_y u) \implies \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = \partial_z u \frac{\partial \boldsymbol{u}}{\partial y} - \partial_y u \frac{\partial \boldsymbol{u}}{\partial z} = 0.$$

If we further assume steady flow, then

$$\frac{D\omega}{Dt} = \boldsymbol{u} \cdot \nabla \omega = 0$$

and we arrive at the following result:

For a steady two-dimensional flow of an ideal fluid subject to a conservative body force, the vorticity ω is constant along a streamline.

An immediate consequence of this is that steady flow past an aerofoil is generally irrotational.

Chapter 3

One-Dimensional Flow

In this chapter we solve various fluid flow problems of a Newtonian fluid, assuming isothermal, incompressible flow and uniform density. This produces the isothermal, incompressible Navier-Stokes equations, which we recall here:

$$\rho\left(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \rho\boldsymbol{g} + \mu\Delta\boldsymbol{u} \tag{3.0.1a}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{3.0.1b}$$

We impose the conditions that all components of the velocity are continuous across a fluid-solid boundary, *i.e.* no-penetration and no-slip boundary conditions. It is possible to introduce the dynamic pressure $P(\boldsymbol{x}, t)$ to include the effect of the body force:

$$-\nabla P = -\nabla p + \rho\boldsymbol{g}. \tag{3.0.2}$$

This implies that the gravity has no effect on the motion and does nothing more than make a contribution to the pressure p as in (3.0.2), but we mention that this is only true if the boundary conditions involve only the velocity [Bat00, Section 4.1]. We will need to use the original expression (3.0.2) later when we discuss in Section 3.3 the steady flow problem down an inclined plane under the influence of gravity.

The main difficulty in solving (3.0.1) is due to the nonlinear advection term $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$. However, this equals zero in certain special cases and it is sometimes possible to find an exact solution because the problem becomes linear. Among the simplest of such special cases are the **unidirectional flow** in which the direction of motion is independent of the position. Consider $\boldsymbol{u}(\boldsymbol{x}, t) = u(\boldsymbol{x}, t)\boldsymbol{e}_1 = u(x, y, z, t)\boldsymbol{e}_1$. By virtue of the incompressibility condition we have

$$\partial_x u = 0 \implies \boldsymbol{u} = u(y, z, t)\boldsymbol{e}_1.$$

We verify that $\boldsymbol{u}\cdot\nabla\boldsymbol{u} \equiv \boldsymbol{0}$:

$$\begin{aligned}
\boldsymbol{u}\cdot\nabla\boldsymbol{u} = (u_i\boldsymbol{e}_i)\cdot(\partial_j u_k\boldsymbol{e}_j\boldsymbol{e}_k) &= u_i\partial_j u_k\boldsymbol{e}_i\cdot\boldsymbol{e}_j\boldsymbol{e}_k \\
&= u_j\partial_j u_k\boldsymbol{e}_k \\
&= w\partial_1 w\boldsymbol{e}_1 \\
&= 0,
\end{aligned}$$

where the third equality follows from $u_2 = u_3 = 0$. Thus, the y- and z-components of the momentum equation reduce to

$$\partial_y P = \partial_z P = 0 \implies P = P(x, t).$$

The x-component of the momentum equation is

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right).$$

Since both the first and last term are independent of x, we deduce that the pressure gradient is a function of time only

$$-\frac{\partial P}{\partial x} = G(t).$$

For positive G, the pressure gradient represents a uniform body force in the direction of $\boldsymbol{e}_1$.

In the case of the steady flow, $\partial_t u = 0$, G is a constant pressure gradient and we obtain the classical two-dimensional Poission equation

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{G}{\mu},$$

with boundary conditions that generally prescribe the pressure gradient $-G$ and the value of u at certain values of y and z. The fluid density ρ is absent since the total acceleration of every fluid element is zero. Each element of the fluid is in equilibrium, so far as the x-components of the forces are concerned, under the action of normal stresses which vary with x (the pressure gradient) and tangential stresses due to viscosity which vary with y and z. In addition, there is a normal stress, hidden by the use of the dynamic pressure P, whose variation with position is such as to make it balance the force of gravity on the element [Bat00, Section 4.2].

3.1 Steady Flow Between Parallel Plane Plate

In the first example, we consider the steady flow between two infinitely long rigid plates that are separated by a distance d. The bottom plate $y = 0$ is stationary while the top plate $y = d$ moves with velocity U in the x-direction. We assume that there is no z-dependence and that the direction of motion is $\boldsymbol{e}_1$, *i.e.* we seek a steady unidirectional flow of the form $\boldsymbol{u} = u(x, y)\boldsymbol{e}_1 = u(y)\boldsymbol{e}_1$, where the x-dependence on u is absent by virtue of the incompressibility condition.

As discussed previously, the problem reduces to a second-order boundary value problem

$$0 = G + \mu u''(y), \quad u(0) = 0, \quad u(d) = U, \tag{3.1.1}$$

and its exact solution takes the form

$$u(y) = -\frac{Gy^2}{2\mu} + \left(\frac{U}{d} + \frac{Gd}{2\mu}\right)y = -\frac{G}{2\mu}y(d-y) + \frac{Uy}{d}. \tag{3.1.2}$$

When the two rigid planes are not in relative motion, the flow is a **plane Poiseuille flow** with a symmetric parabolic velocity profile

$$u(y) = -\frac{G}{2\mu}y(d-y). \tag{3.1.3}$$

When the pressure gradient G is zero, the flow is a simple shear flow, called **plane Couette flow**, with a linear velocity profile

$$u(y) = \frac{Uy}{d}. \tag{3.1.4}$$

A notable aspect of the plane Couette flow is that shear stress $\partial_y u$ is constant throughout the flow domain.

The previous analysis suggests that the flow behaves like a plane Couette (or Poiseuille) flow if the pressure gradient G (or the speed of the plate U) is sufficiently small, but small *compared with what*? Because the problem has four parameters G, μ, d, U and they involve units, we non-dimensionalise the problem so that we can compare the magnitude of terms appearing in the solution with just numbers.

1. Choose a characteristic length scale $\ell_c = d$ and characteristic velocity $v_c = U$ and define the dimensionless variables

$$\tilde{y} = \frac{y}{\ell_c} = \frac{y}{d}, \quad \tilde{u} = \frac{u}{v_c} = \frac{u}{U}.$$

It follows from chain rule that

$$\frac{d}{dy} = \frac{1}{d}\frac{d}{d\tilde{y}}$$

and we obtain the dimensionless equation

$$\begin{aligned} \frac{d^2 u}{dy^2} &= -\frac{G}{\mu} \\ \frac{U}{d^2}\frac{d^2\tilde{u}}{d\tilde{y}^2} &= -\frac{G}{\mu} \\ \frac{d^2\tilde{u}}{d\tilde{y}^2} &= -\frac{Gd^2}{\mu U} = -K \end{aligned}$$

with boundary condition $\tilde{u}(0) = 0$ and $\tilde{u}(1) = 1$. It should be noted that K is a dimensionless parameter. The exact solution is

$$\tilde{u}(\tilde{y}) = -\frac{K}{2}\left(\tilde{y}^2 - \tilde{y}\right) + \tilde{y}.$$

It is evident that the flow behaves like a plane Couette flow if $K \ll 1$, *i.e.* $G \ll \mu U/d^2$.

2. On the other hand, the solution blows up for the other limiting case $K \gg 1$ and this indicates that we need to choose the characteristic scales differently. The issue is that the terms do not balance as K grows large in the dimensionless equation

$$\frac{d^2\tilde{u}}{d\tilde{y}^2} = -K.$$

To resolve this issue, we define a new dimensionless velocity variable

$$\bar{u} = \frac{\tilde{u}}{K} = \frac{u}{UK} = \frac{u}{\left(\dfrac{Gd^2}{\mu}\right)}.$$

which amounts to choosing the characteristic velocity v_c as Gd^2/μ. The new dimensionless problem is

$$\frac{d^2\bar{u}}{d\tilde{y}^2} = -1, \quad \bar{u}(0) = 0, \quad \bar{u}(1) = \frac{\mu U}{Gd^2} = \frac{1}{K}$$

and its exact solution is

$$\bar{u}(\tilde{y}) = -\frac{\tilde{y}^2}{2} + \left(\frac{1}{K} + \frac{1}{2}\right)\tilde{y}.$$

We see that the flow behaves like a plane Poiseuille flow if $K \gg 1$, *i.e.* $U \ll Gd^2/\mu$.

3.2 Rayleigh Problem

This is a problem concerning the flow created by a sudden movement of a plane from rest. There are a few variants of the problem, such as infinite vs finite domain and constant speed plane motion vs time-dependent plane motion.

3.2.1 Infinite domain - constant speed

Consider a viscous fluid in the infinite domain $y > 0$ and suppose that at $t = 0$ the plate $y = 0$ is suddenly jerked into motion in the x-direction with constant speed U. It is natural to look for time-dependent unidirectional flow $\boldsymbol{u} = u(y,t)\boldsymbol{e}_1$ similar to the parallel plate problem. We assume that there is no pressure gradient between $x = \pm\infty$, *i.e.* the flow is only due to the motion of the plate. Define the **kinematic viscosity** $\nu = \mu/\rho$, the x-component of the momentum equation reduce to the classical one-dimensional diffusion equation

$$\partial_t u = \nu \partial_{yy} u, \tag{3.2.1}$$

with initial condition $u(y,0) = 0$ for $y > 0$ and boundary conditions

$$u(0,t) = U \quad \text{and} \quad u(\infty,t) = 0 \quad \text{for } t > 0.$$

We employ a technique, called **similarity transformation**, to solve the problem, but before that let us explain why this technique is applicable here. Observe that there is no obvious length scale ℓ_c if we attempt to non-dimensionalise the problem but the solution depends on time. It is then necessary to construct both the length and time scale using the independent variables y, t and parameters ν, U. These constructions are not "guessed"', rather they are derived from the scaling of the governing equations. Indeed, a dimensional analysis on the governing equation yields

$$\frac{U}{t} \sim \nu\left(\frac{U}{y^2}\right) \implies y \sim \sqrt{\nu t}.$$

Instead, we invoke a common approach from the theory of linear PDEs, where one looks for symmetry transformation (scale invariance) of the problem, that is, a dilation transformation of the form

$$y \mapsto \tilde{y} = ay, \quad t \mapsto \tilde{t} = a^\beta t,$$

such that the governing equation for $u(\tilde{y},\tilde{t})$ does not change. Chain rule gives

$$\frac{\partial}{\partial y} = a\frac{\partial}{\partial \tilde{y}} \quad \text{and} \quad \frac{\partial}{\partial t} = a^\beta \frac{\partial}{\partial \tilde{t}},$$

and the equation for $u(\tilde{y},\tilde{t})$ is

$$a^\beta \partial_{\tilde{t}} u = \nu a^2 \partial_{\tilde{y}\tilde{y}} u.$$

Consequently, if $u(y,t)$ is a solution of (3.2.1), then $u(ay, a^2 t)$ is also a solution of (3.2.1) for any constant $a \in \mathbb{R}$. More generally, If a given PDE in $u(y,t)$ has a symmetry transformation $(y,t) \mapsto (ay, a^\beta t)$ for some β, then a solution of the PDE is given by $u = f(\eta)$, where $\eta = y/t^{1/\beta}$.

f is called the **similarity transformation** and η the **similarity variable**. To this end, let $\eta = y/\sqrt{t}$ and assume that $u = f(\eta)$ for some function f. Then

$$\begin{aligned}
\partial_t u &= f'(\eta)\partial_t \eta = f'(\eta)\left(-\frac{y}{2t^{3/2}}\right) = -f'(\eta)\left(\frac{\eta}{2t}\right) \\
\partial_y u &= f'(\eta)\partial_y \eta = f'(\eta)\left(\frac{1}{\sqrt{t}}\right) \\
\partial_{yy} u &= \partial_y(\partial_y u) = f''(\eta)\left(\frac{1}{t}\right)
\end{aligned}$$

and so the governing equation (3.2.1) becomes

$$-f'(\eta)\left(\frac{\eta}{2t}\right) = \nu\left(\frac{f''(\eta)}{t}\right) \implies f''(\eta) + \left(\frac{\eta}{2\nu}\right) f'(\eta) = 0. \tag{3.2.2}$$

The boundary conditions are

$$\begin{aligned}
U = u(0,t) &= f(\eta)|_{y=0} = f(0) \\
0 = u(\infty,t) &= f(\eta)|_{y=\infty} = f(\infty).
\end{aligned}$$

Note that the condition $f(\infty) = 0$ coincides with the initial condition as well. Define $g(\eta) = f'(\eta)$, then (3.2.2) becomes

$$g'(\eta) + \left(\frac{\eta}{2\nu}\right) g(\eta) = 0. \tag{3.2.3}$$

The method of integrating factor can be used to solve (3.2.3) and its the general solution

$$g(\eta) = Be^{-\eta^2/4\nu}.$$

Consequently,

$$f(\eta) = A + B\int_0^\eta e^{-s^2/4\nu}\, ds.$$

The boundary condition $f(0) = U$ gives $A = U$, and the far field boundary condition implies

$$\begin{aligned}
0 &= A + B\int_0^\infty e^{-s^2/4\nu}\, ds \\
&= U + 2\sqrt{\nu}B\int_0^\infty e^{-\tau^2}\, d\tau \\
&= U + \sqrt{\pi\nu}B \\
\implies B &= -\frac{U}{\sqrt{\pi\nu}}.
\end{aligned}$$

Hence, the solution of the problem is

$$u = f(\eta) = U\left[1 - \frac{1}{\sqrt{\pi\nu}}\int_0^\eta e^{-s^2/4}\, ds\right], \quad \text{where } \eta = y/\sqrt{t}. \tag{3.2.4}$$

Another choice of similarity variable, which is motivated by scaling law, is $\eta = y/\sqrt{\eta t}$ which is now dimensionless. A scaling argument on (3.2.4) shows that

$$u = U\left[1 - \frac{1}{\sqrt{\pi}}\int_0^\eta e^{-s^2/4}\,ds\right], \quad \text{where } \eta = y/\sqrt{\nu t}. \tag{3.2.5}$$

These two solutions are the same, but the latter is preferrable since the similarty variable $\eta = y/\sqrt{\eta t}$ is dimensionless.

Remark 3.2.1. There are necessary conditions for the existence of similarity solution:

1. The similarity variable η may depends on the parameters and independent variables, but the original PDE for u must reduce to an ODE for f as a function of η only.
2. The original initial and boundary conditions must reduce to the appropriate number and type of conditions on F, so that this is consistent with the order of ODE for f.

It should be noted that the simple form of the initial and boundary conditions was essential to the success of the method. Let us interpret the solution (3.2.5). At any two different times t_1, t_2, the velocity u is the same function of $y/\sqrt{\nu t}$. What happens is that the velocity profile becomes stretched out as time flows.

3.2.2 Finite domain - constant speed

Consider the same configuration as in the previous subsection, but on a finite domain $0 < y < d$. The problem remains the same, the only difference being we replace the far-field condition with the no-slip condition on the top plate $y = d$:

$$\partial_t u = \nu\partial_{yy}u \quad \text{for } 0 < y < d \tag{3.2.6a}$$
$$u(y,0) = 0 \quad \text{for } 0 < y < d \tag{3.2.6b}$$
$$u(0,t) = U \quad \text{for } t > 0 \tag{3.2.6c}$$
$$u(d,t) = 0 \quad \text{for } t > 0. \tag{3.2.6d}$$

Since the dimensional parameter d enters the problem, $y/\sqrt{\nu t}$ is no longer the only dimensionless combination of the available parameters, and we have no grounds for anticipating a similarity solution.

Separation of variables is not applicable at first sight since the bottom boundary condition is not homogeneous. A general technique to resolve this issue is to decompose $u(y,t)$ into its equilibrium solution $u_s(y)$ plus some other function $w(y,t)$, *i.e.*

$$u(y,t) = u_s(y) + w(y,t), \tag{3.2.7}$$

We may infer from Section 3.1 that

$$u_s(y) = U\left(1 - \frac{y}{d}\right). \tag{3.2.8}$$

Substituing (3.2.7) and (3.2.9) into (3.2.6) then yields a new problem in terms of $w(y,t)$:

$$\partial_t w = \nu\partial_{yy}w \quad \text{for } 0 < y < d \tag{3.2.9a}$$

$$w(y,0) = -u_s(y) = -U\left(1 - \frac{y}{d}\right) \quad \text{for } 0 < y < d \tag{3.2.9b}$$

$$w(0,t) = U - u_s(0) = 0 \quad \text{for } t > 0 \tag{3.2.9c}$$

$$w(d,t) = 0 - u_s(0) = 0 \quad \text{for } t > 0. \tag{3.2.9d}$$

This new problem now has homogeneous boundary conditions, but the tradeoff is that the initial condition might gets complicated.

Assuming an ansatz of the form $w(y,t) = f(y)g(t)$, then the governing equation of (3.2.9) becomes

$$g'(t)f(y) = \nu f''(y)g(t) \implies \frac{g'(t)}{g(t)} = \nu\left(\frac{f''(y)}{f(y)}\right) = -\nu\lambda^2.$$

Solving the ODE for $f(y)$ yields the general solution

$$f(y) = A\cos(\lambda y) + B\sin(\lambda y).$$

The boundary condition $f(0) = 0$ gives $A = 0$, and the other boundary condition implies

$$0 = f(d) = B\sin(\lambda d) \implies \lambda = \frac{n\pi}{d}.$$

Therefore

$$f(y) = B_n \sin\left(\frac{n\pi y}{d}\right).$$

On the other hand, solving the ODE for $g(t)$ yields

$$g(t) = Ce^{-\nu\lambda^2 t} = C\exp\left(-\frac{n^2\pi^2\nu t}{d^2}\right).$$

Hence,

$$w(y,t) = \sum_{n=1}^{\infty} A_n \exp\left(-\frac{n^2\pi^2\nu t}{d^2}\right)\sin\left(\frac{n\pi y}{d}\right),$$

where the coefficients A_n are given by

$$A_n = -\frac{2}{d}\int_0^d U\left(1 - \frac{y}{d}\right)\sin\left(\frac{n\pi y}{d}\right)\,dy = -\frac{2U}{n\pi}.$$

Finally, the solution to the original problem (3.2.6) is

$$u(y,t) = U\left(1 - \frac{y}{d}\right) - \sum_{n=1}^{\infty}\left(\frac{2U}{n\pi}\right)\exp\left(-\frac{n^2\pi^2\nu t}{d^2}\right)\sin\left(\frac{n\pi y}{d}\right).$$

3.2.3 Oscillating plane boundary

3.3 Steady Flow Down an Inclined Plane

3.4 Taylor-Couette flow

Remark 3.4.1. Consider two rotating long concentric cylinders, and there is some fluid in between these cylinders. This shares the same structure as the previous example, except that we are solving the problem in cylindrical coordinate.

Chapter 4

Surface Waves

A classical model for the small transversal vibrations $h(x,t)$ of a tightly stretched horizontal string is the one-dimensional wave equation

$$h_{tt} = c^2 h_{xx}, \quad \text{where } c^2 = \frac{T}{\rho}$$
$$T = \text{tension force}$$
$$\rho = \text{linear density of the string at rest.}$$

Consider the Fourier transform with respect to x, for each fixed t, of $h(x,t)$:

$$\hat{h}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x,t) e^{-ikx}\, dx.$$

Applying the Fourier transform to the wave equation yields

$$\hat{h}_{tt} = -c^2 k^2 \hat{h}$$

which has general solution

$$\hat{h}(k,t) = A(k) e^{-ickt} + B(k) e^{ickt}.$$

We take the inverse Fourier transform to recover $h(x,t)$:

$$\begin{aligned}
h(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{h}(k,t) e^{ikx}\, dk \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[A(k) e^{-ickt} e^{ikx} + B(k) e^{ickt} e^{ikx} \right] dk \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[A(k) e^{ik(x-ct)} + B(k) e^{ik(x+ct)} \right] dk \\
&= F(x-ct) + G(x-ct).
\end{aligned}$$

The functions $A(k), B(k)$ are known as the Fourier component which are determined by initial and boundary conditions. Here, k is the wavenumber and it relates to the wavelength $\lambda = 2\pi/k$. The quantity $\omega = ck$ is the temporal frequency and it relates to the period $T = 2\pi/\omega$. The expression for $h(x,t)$ represents the superposition of two travelling waves moving at constant speed $c > 0$ in the positive and negative $x-$ direction respectively and these waves are not dispersive, *i.e.* all disturbances travel at a constant speed.

In contrast with these small amplitude waves on a taut string, water waves are dispersive, *i.e.* different Fourier components that make up a general disturbance travels at different speeds, depending on their wavelength. Water waves problem are very different than the fixed-boundary problems considered in Chapter 3, in the sense that the water surface, called the **free surface**, is part of the boundary and it varies in time. This means that we need to solve for both the velocity field $\boldsymbol{u}$ and the moving free surface. This belongs to a more general class of problem known as **free boundary problem**.

4.1 Surface Waves on Deep Water

Let us begin with the two-dimensional water waves problem. Consider an irrotational flow of an incompressible, inviscid fluid occupying the domain $\mathcal{D}_t$ with infinite depth, defined by

$$\mathcal{D}_t = \{(x, y) \in \mathbb{R}^2 \colon -\infty < x < \infty, -\infty < y < \eta(x, t)\}.$$

We assume that the mean free surface is located at $y = 0$ and the interface between fluid and air, *i.e.* the free surface, is the graph of some unknown displacement function $y = \eta(x, t)$.

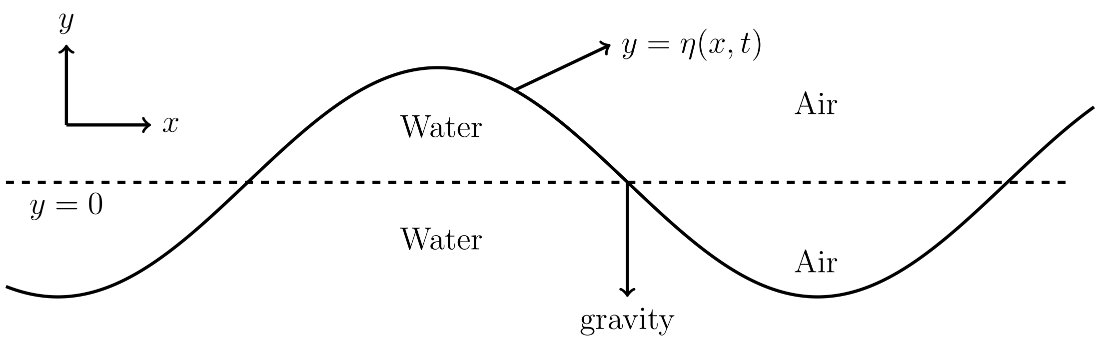

Figure 4.1: Surface waves on deep water.

4.1.1 Existence of velocity potential

Since the fluid is assumed to be inviscid, it follows from the two-dimensional vorticity equation that

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = \boldsymbol{0}$$

and the vorticity $\boldsymbol{\omega} = (0, 0, \omega)$ of each fluid element remains unchanged. In particular, $\omega \equiv 0$ since the flow is assumed to be irrotational. Consequently, there exists a function $\phi(x, y, t)$, called the **velocity potential**, such that $\boldsymbol{u} = \nabla\phi$. We verify that ϕ satisfies the zero vorticity condition:

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial \phi}{\partial x} = 0.$$

The existence of velocity potential is a result from multivariable calculus:

Any C^1 vector field $\boldsymbol{u}$ on a simply-connected domain $\mathcal{D} \subset \mathbb{R}^3$ is conservative if and only if it is irrotational on $\mathcal{D}$.

We point out that the *if statement* is false if U is not simply-connected. For completeness, we prove the result in the two-dimensional case. Fix a point (x_0, y_0) in $\mathcal{D}$ and define the function $\phi(x, y, t) : \mathcal{D} \longrightarrow \mathbb{R}$ as the line integral

$$\phi(x,y,t) = \int_{(x_0,y_0)}^{(x,y)} u(x',y',t)\,dx' + v(x',y',t)\,dy'.$$

Since path-independence of the line integral is equivalent to the vector field $\boldsymbol{u}$ being conservative, it suffices to show that ϕ is path-independent. Let C_1, C_2 be two different paths from (x_0, y_0) to (x, y). By construction, the curve $C_1 \cup (-C_2)$ encloses a region S in $\mathcal{D}$. Since $\mathcal{D}$ is simply-connected, we may apply Stokes' theorem and find that

$$\begin{aligned}\oint_{C_1\cup(-C_2)} u\,dx' + v\,dy' &= \oint_{P_1\cup(-P_2)} \boldsymbol{u}\cdot\boldsymbol{t}\,ds \\ &= \int_S (\nabla\times\boldsymbol{u})\cdot\boldsymbol{n}\,dS \\ &= \int_S (\nabla\times\boldsymbol{u})\,dA \\ &= \int_S \omega\boldsymbol{e}_3\,dA = 0.\end{aligned}$$

By virtue of the incompressibility condition the velocity potential ϕ satisfies Laplace's equation in $\mathcal{D}_t$:

$$0 = \nabla\cdot\boldsymbol{u} = \nabla\cdot(\nabla\phi) = \Delta\phi.$$

4.1.2 Kinematic boundary condition

In the water-waves problem, water and air are two immiscible fluids since they do not mix into each together. Let us define the free surface as the zero level set of the implicit function

$$F(x,y,t) = y - \eta(x,t).$$

The no-penetration boundary condition for a moving boundary is that

$$\boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{u}_{\text{moving boundary}}\cdot\boldsymbol{n}.$$

This condition is equivalent to the requirement that *fluid particles on the free surface must remain on the free surface*, *i.e.* the material derivative of F must be zero:

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \nabla F\cdot\boldsymbol{u} = 0. \tag{4.1.1}$$

This is called the kinematic boundary condition because it only involves the flow field. Rewriting (4.1.1) using $\nabla F = \boldsymbol{n}|\nabla F|$, where $\boldsymbol{n}$ is the unit outward normal (pointing in the $\hat{\boldsymbol{y}}$ direction), we obtain

$$\begin{aligned} 0 &= \frac{\partial F}{\partial t} + |\nabla F|\boldsymbol{n}\cdot\boldsymbol{u} \\ 0 &= \frac{1}{|\nabla F|}\frac{\partial F}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{n}. \end{aligned}$$

In particular, this asserts that the zero level set of F is convected in the normal direction $\boldsymbol{n}$ with the flow field $\boldsymbol{u}$. Finally, computing and substituting all the derivatives of F into (4.1.1) yields the kinematic equation for the free surface:

$$\eta_t + u\eta_x = v \quad \text{on } y = \eta(x,t).$$

4.1.3 Dynamic boundary condition

For an irrotational flow of an ideal fluid with gravitational body force $\boldsymbol{F}_b = \boldsymbol{g} = -g\hat{\boldsymbol{y}}$, the Cauchy momentum equation (see (2.7.1) in the derivation of Bernoulli streamline theorem) reduces to

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla\left(\frac{p}{\rho} + gy + \frac{1}{2}|\boldsymbol{u}|^2\right) \quad \text{on } y \le \eta(x,t) \tag{4.1.2}$$

Using $\boldsymbol{u} = \nabla\phi$ and integrating (4.1.2) with respect to $\boldsymbol{x}$ gives

$$\phi_t + \frac{p}{\rho} + gy + \frac{1}{2}|\nabla\phi|^2 = G(t) \quad \text{on } y \le \eta(x,t) \tag{4.1.3}$$

where $G(t)$ is an arbitrary function of time t. In particular, $G(t)$ may be chosen arbitrarily by absorbing a suitable function of t into ϕ using the transformation

$$\Phi(x,y,t) = \phi(x,y,t) - \int_{t_0}^{t} G(\tau)\, d\tau \tag{4.1.4}$$

in which (4.1.3) reduces to

$$\Phi_t + \frac{p}{\rho} + gy + \frac{1}{2}|\nabla\Phi|^2 = 0 \quad \text{on } y \le \eta(x,t). \tag{4.1.5}$$

This does not affect the relation of ϕ to the flow velocity since $\boldsymbol{u} = \nabla\phi = \nabla\Phi$. We remark that (4.1.3) is the **Bernoulli equation for unsteady potential flow** and it was used in the derivation of Luke's variational principle which is a Lagrangian variational description of the free surface motion under the influence of gravity [Luk67].

Since the flow is inviscid, there are no tangential stresses and the tangential stress is balanced at the free surface, provided we neglect the surface tension. On the other hand, since the only normal stress exerted on the free surface is the pressure, the normal stress boundary condition translates to no pressure jump across the free surface, *i.e.*

$$p(x, \eta(x,t)^-, t) = p_0 \quad \text{at } y = \eta(x,t)$$

with p_0 the (uniform) atmospheric pressure. By virtue of (4.1.4), evaluating (4.1.3) at $y = \eta(x,t)$ and choosing $G(t) = p_0/\rho$ to eliminate the constant term we obtain the dynamic boundary condition:

$$\phi_t + g\eta + \frac{1}{2}|\nabla\phi|^2 = 0 \quad \text{on } y = \eta(x,t).$$

4.1.4 Linearisation of the surface boundary condition

The two-dimensional water-waves problem, in terms of the velocity potential ϕ, takes the form

$$\begin{aligned} \Delta\phi &= 0 && \text{for } y < \eta(x,t) \\ \eta_t + \phi_x\eta_x &= \phi_y && \text{on } y = \eta(x,t) \\ \phi_t + g\eta + \frac{1}{2}|\nabla\phi|^2 &= 0 && \text{on } y = \eta(x,t). \end{aligned}$$

This problem is difficult to solve in general since the boundary conditions are nonlinear. Assuming the free surface displacement $\eta(x,t)$ and the fluid velocities u, v are small, in a sense to be made precise later. This allows us to linearise the problem by dropping quadratic (and higher terms) of u, v, η and the two surface conditions on $y = \eta(x,t)$ reduce to

$$\begin{aligned} \eta_t &= \phi_y = v \\ \phi_t + g\eta &= 0. \end{aligned}$$

Expanding $v(x,\eta,t)$ and $\phi(x,\eta,t)$ around the mean free surface $y = 0$ and neglecting quadratic (and higher terms) again, we obtain

$$\begin{aligned} \eta_t(x,t) &= v(x,0,t) \\ \phi_t(x,0,t) + g\eta(x,t) &= 0. \end{aligned}$$

This allows us to rewrite the surface boundary conditions at $y = \eta(x,t)$ as surface boundary conditions on $y = 0$. Consequently, we obtain the linear water-waves problem:

$$\Delta\phi = 0 \qquad \text{for } y < \eta(x,t) \tag{4.1.6a}$$

$$\eta_t = \phi_y \qquad \text{on } y = 0 \tag{4.1.6b}$$

$$\phi_t + g\eta = 0 \qquad \text{on } y = 0. \tag{4.1.6c}$$

4.1.5 Travelling wave solution

We look for sinusoidal travelling wave solution for the free surface displacement

$$\eta(x,t) = A\cos(kx - \omega t), \quad \text{where } \begin{aligned} A &= \text{amplitude} \\ k &= \text{wavenumber} \\ \omega &= \text{angular frequency} \\ \lambda &= \frac{2\pi}{k} = \text{spatial wavelength} \\ T &= \frac{2\pi}{\omega} = \text{temporal period.} \end{aligned}$$

WLOG, the phase shift can be chosen to be zero by appropriately translating t, which is the case here. The boundary condition (4.1.6b) and (4.1.6c) suggest that the velocity potential should be of the form

$$\phi(x,y,t) = f(y)\sin(kx - \omega t).$$

For this ansatz of ϕ to satisfy Laplace's equation (4.1.6a), the function $f(y)$ must satisfy

$$f''(y) - k^2 f(y) = 0 \quad \text{for } y < \eta(x,t)$$

which has general solution

$$f(y) = Ce^{ky} + De^{-ky}.$$

WLOG, we may take $k > 0$. Since the water is of infinite depth, we require the velocity to be bounded as $y \longrightarrow -\infty$ and this is not possible unless $D = 0$. Thus

$$\phi(x,y,t) = Ce^{ky}\sin(kx - \omega t).$$

Substituting this, together with the ansatz of η, into both (4.1.6b) and (4.1.6c) and cancelling the common sinusoidal term, we obtain

$$\begin{cases} \omega A - kC &= 0 \\ -\omega C + gA &= 0 \end{cases} \iff \begin{bmatrix} \omega & -k \\ g & -\omega \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This system has a nontrivial solution if and only if the matrix is nonsingular, that is if

$$\omega^2 = gk. \tag{4.1.7}$$

This equation relating the frequency ω and the wavenumber k is called the **dispersion relation**. It takes the form $\omega^2 = g|k|$ if no restriction is placed on the sign of k. For any given $A > 0$ the solution of the linear water-waves problem (4.1.6) is

$$\phi(x, y, t) = \frac{A\omega}{k} e^{ky} \sin(kx - \omega t) \tag{4.1.8a}$$
$$\eta(x, t) = A\cos(kx - \omega t) \tag{4.1.8b}$$
$$\omega^2 = gk. \tag{4.1.8c}$$

The dynamic pressure P for the linear water-waves can be recovered from Bernoulli equation for unsteady potential flow (4.1.5). Neglecting higher-order terms we find that

$$\begin{aligned} P = p + \rho g y = -\rho\phi_t &= \rho \frac{A\omega^2}{k} e^{ky} \cos(kx - \omega t) \\ &= \rho g A e^{ky} \cos(kx - \omega t). \end{aligned}$$

For $k > 0$, the surface waves travel to the right with phase velocity

$$c_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}$$

and they are dispersive since waves with different wavenumbers move at different speed. Moreover, the phase velocity is an increasing function of the wavelength λ so longer waves propagates faster.

4.1.6 Small amplitude assumption

The linear water-wave problem is obtained by assuming η, u, v are small, but small compared to what? The standard procedure to compare magnitudes of different terms is to nondimensionalise, but here we take advantage of the fact that we have the explicit solution. The expression for ϕ shows that that u and v are of the same order of magnitude. To obtain (4.1.6b), we neglected the term $u\eta_x$ compared with the term v. This is reasonable provided

$$|u\eta_x| \ll |v| \iff |\eta_x| \ll \iff |Ak| \ll 1 \iff |A| \ll \lambda.$$

Thus, we require that the free surface displacement is small compared to the wavelength of the waves and this is sometimes called the **small-slope approximation**. To obtain (4.1.6c), we neglected the term $|\nabla\phi|^2 = u^2 + v^2$ compared with the term $g\eta$. This is reasonable provided

$$|u^2 + v^2| \ll g|\eta| \iff A^2\omega^2 \ll gA \iff A^2 gk \ll gA \iff |Ak| \ll 1$$

and we again recover the small-slope approximation.

4.1.7 Particle paths

From (4.1.8c) the velocity components are

$$u(x,y,t) = \phi_x = A\omega e^{ky}\cos(kx - \omega t)$$
$$v(x,y,t) = \phi_y = A\omega e^{ky}\sin(kx - \omega t).$$

At a particular time $t > 0$, a fluid particle initially at its mean position $(\bar{x}, \bar{y})$ reaches (x, y) and its particle velocity at that point must satisfy

$$\frac{dx}{dt} = u(x,y,t) \quad \text{and} \quad \frac{dy}{dt} = v(x,y,t)$$

and in principle we can integrate these to obtain the particle path. In the linear water-waves problem, any particle deviates only a small amount from its mean position, *i.e.*

$$x(t) = \bar{x} + \tilde{x}(t)$$
$$y(t) = \bar{y} + \tilde{y}(t).$$

Since $\tilde{x}(t) \ll x(t)$ and $\tilde{y}(t) \ll y(t)$, we can approximate the RHS of the ODEs as

$$\frac{d\tilde{x}}{dt} = u(\bar{x},\bar{y},t) = A\omega e^{k\bar{y}}\cos(k\bar{x} - \omega t)$$
$$\frac{d\tilde{y}}{dt} = v(\bar{x},\bar{y},t) = A\omega e^{k\bar{y}}\sin(k\bar{x} - \omega t)$$

which can be easily integrated. Hence

$$\tilde{x}(t) = -Ae^{k\bar{y}}\sin(k\bar{x} - \omega t)$$
$$\tilde{y}(t) = Ae^{k\bar{y}}\cos(k\bar{x} - \omega t)$$

and the particle paths are circular with radius $Ae^{k\bar{y}}$ since

$$\tilde{x}^2 + \tilde{y}^2 = \left(Ae^{k\bar{y}}\right)^2.$$

In particular, the radius and the fluid velocities both decrease exponentially with the depth and so the fluid motion is limited to the free surface of depth $\bar{y} = \mathcal{O}(1/k) = \mathcal{O}(\lambda)$.

4.2 Group Velocity

As a motivating example, consider the superposition of travelling sinusoidal waves

4.3 Surface Tension Effects: Capillary Waves

4.4 Nonlinear shallow water equations

4.4.1 Method of characteristcs

4.4.2 Dam breaking

4.4.3 Bore

4.5 Lubrication Theory

$$\frac{\partial}{\partial y}\Big[(1+\varepsilon y)u_r\Big] + \frac{\partial}{\partial \theta}u_\theta = 0$$

$$\left(\frac{\rho a^2\Omega}{\mu}\right)\varepsilon\left\{\varepsilon\left(u_r\frac{\partial u_r}{\partial y} + \frac{u_\theta}{1+\varepsilon y}\frac{\partial u_r}{\partial \theta}\right) - \frac{u_\theta^2}{(1+\varepsilon y)^2}\right\}$$

$$= -\frac{P_c}{\mu\Omega}\frac{\partial P}{\partial y} + \left\{\frac{1}{1+\varepsilon y}\frac{\partial}{\partial y}\left[(1+\varepsilon y)\frac{\partial u_r}{\partial y}\right] + \frac{\varepsilon^2}{(1+\varepsilon y)^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{\varepsilon^2}{(1+\varepsilon y)^2}u_r - \frac{2\varepsilon}{(1+\varepsilon y)^2}\frac{\partial u_\theta}{\partial \theta}\right\}$$

$$\left(\frac{\rho a^2\Omega}{\mu}\right)\varepsilon^2\left\{u_r\frac{\partial u_\theta}{\partial y} + \frac{u_\theta}{1+\varepsilon y}\frac{\partial u_\theta}{\partial \theta} + \frac{\varepsilon u_\theta u_r}{1+\varepsilon y}\right\}$$

$$= -\frac{P_c\varepsilon^2}{\mu\Omega}\frac{1}{1+\varepsilon y}\frac{\partial P}{\partial \theta} + \left\{\frac{1}{1+\varepsilon y}\frac{\partial}{\partial y}\left[(1+\varepsilon y)\frac{\partial u_\theta}{\partial y}\right] + \frac{\varepsilon^2}{(1+\varepsilon y)^2}\left(\frac{\partial^2 u_\theta}{\partial \theta^2} - u_\theta\right) + \frac{2\varepsilon^2}{(1+\varepsilon y)^2}\frac{\partial u_r}{\partial \theta}\right\}$$

The boundary conditions becomes $u_r = 0, u_\theta = -1$ at $y = 0$ and $u_r = u_\theta = 0$ at $y = h(\theta) = 1 + \lambda\cos(\theta)$. The Reynolds number in this case is

$$\mathrm{Re} = \frac{\rho a^2\Omega}{\mu}.$$

and $\mathrm{Re}\varepsilon \ll 1$ since $\varepsilon \ll 1$, assuming $\mathrm{Re} = \mathcal{O}(1)$. Let us try a regular asymptotic expansion

$$\begin{aligned} u_r &= u_r^0 + \mathcal{O}(\varepsilon, \varepsilon\mathrm{Re}) \\ u_\theta &= u_\theta^0 + \mathcal{O}(\varepsilon, \varepsilon\mathrm{Re}) \\ P &= P^0 + \mathcal{O}(\varepsilon, \varepsilon\mathrm{Re}) \end{aligned}$$

Collecting only $\mathcal{O}(1)$ terms, we get the first order lubrication equations

$$\frac{\partial u_r^0}{\partial y} + \frac{\partial u_\theta^0}{\partial \theta} = 0$$

$$0 = -\frac{P_c}{\mu\Omega}\frac{\partial P^0}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial u_r^0}{\partial r}\right)$$

$$0 = -\frac{P_c\varepsilon^2}{\mu\Omega}\frac{\partial P^0}{\partial \theta} + \frac{\partial^2}{\partial y^2}u_\theta^2.$$

Recall that for the concentric circle (Coutte flow), we have

$$\frac{u_\theta^2}{y} = \frac{\partial P}{\partial y}$$

$$0 = \frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial u_\theta}{\partial y}\right) - \frac{u_\theta}{y^2}.$$

Centripetal acceleration u_θ^2/y balanced by the pressure gradient. On the lubrication ones, the pressure gradient is balanced by the viscous term.

Chapter 5

Classical Aerofoil Theory

5.1 Velocity Potential and Stream Function

Given an irrotational vector field $\boldsymbol{u}$ defined on some fluid region $\mathcal{D}$, we may define a potential $\phi\colon \mathcal{D} \longrightarrow \mathbb{R}$ as the line integral

$$\phi(\boldsymbol{P}) = \int_{\boldsymbol{O}}^{\boldsymbol{P}} \boldsymbol{u} \cdot d\boldsymbol{x},$$

where $\boldsymbol{O} \in \mathcal{D}$ is some arbitrary reference point. For simply-connected fluid region $\mathcal{D}$, the potential ϕ is independent of the path between $\boldsymbol{O}$ and $\boldsymbol{P}$. Consequently, ϕ is a well-defined function and we have $\boldsymbol{u} = \nabla\phi$.

Example 5.1.1. Consider a rigid body rotation with angular velocity Ω. The flow field is given by

$$\boldsymbol{u} = (0, 0, \Omega) \times (x, y, z) = (-\Omega y, \Omega x, 0)$$

and

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = 2(0, 0, \Omega) = 2\boldsymbol{\Omega} \neq \boldsymbol{0}.$$

Example 5.1.2. Consider the two-dimensional irrotational vortex with $\boldsymbol{\Omega} = (0, 0, \alpha/r^2)$. Then

$$\boldsymbol{u} = \Big(-\frac{\alpha y}{r^2}, \frac{\alpha x}{r^2}, 0\Big)$$

and one can verify that $\boldsymbol{\omega} = \boldsymbol{0}$.

Example 5.1.3. Consider the stagnation point flow

$$\boldsymbol{u} = (\alpha x, -\alpha y, 0).$$

One can verify that $\boldsymbol{\omega} = \boldsymbol{0}$ and there exists a velocity potential ϕ such that $\boldsymbol{u} = \nabla\phi$, with

$$\phi(x, y) = \frac{\alpha}{2}(x^2 + y^2),$$

up to some additive constant.

Example 5.1.4. Consider the line vortex flow $\boldsymbol{u} = \frac{k}{r}\boldsymbol{e}_\theta$, with $\boldsymbol{e}_\theta$ the unit vector in polar coordinates defined by

$$\boldsymbol{e}_\theta = (-\sin\theta, \cos\theta, 0).$$

Using the gradient operator in polar coordinates, the velocity potential ϕ, if it exists, satisfies

$$\partial_r\phi = 0, \quad \frac{1}{r}\partial_\theta\phi = \frac{k}{r}, \quad \partial_z\phi = 0.$$

Up to some additive constant, we find that $\phi = k\theta$. This doesn't contradict the fact we mentioned above since $\boldsymbol{u}$ is defined on $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ which is not simply-connected. For closed curves that don't go around the origin, the line integral is zero. But for the unit circle centered at the origin, the line integral is

$$\oint_C \boldsymbol{u} \cdot d\boldsymbol{x} = 2\pi k.$$

Definition 5.1.5. For any closed, smooth, oriented, simple curve C, the circulation of $\boldsymbol{u}$ around C is defined as

$$\Gamma = \oint_C \boldsymbol{u} \cdot d\boldsymbol{x}.$$

If the region S enclosed by the curve C is simply-connected, it follows from Stokes theorem that

$$\Gamma = \iint_S \boldsymbol{\omega} \cdot \boldsymbol{n}\, dS.$$

Definition 5.1.6. A potential flow is a flow that can be represented as $\boldsymbol{u} = \nabla\phi$.

If $\boldsymbol{u}$ is irrotational and incompressible, then the corresponding velocity potential ϕ must be harmonic since

$$\nabla \cdot \boldsymbol{u} = \nabla \cdot \nabla\phi = \Delta\phi = 0.$$

For the remaining section, we are only considering two-dimensional incompressible flow.

Definition 5.1.7. A function ψ is a **stream function** of $\boldsymbol{u} = (u, v)$ if

$$u = \partial_y\psi \quad \text{and} \quad v = -\partial_x\psi.$$

ψ is called the stream function because it is constant along any streamlines. Note that by construction the incompressibility condition is automatically satisfied too.

Let us try to relate the stream function and the velocity potential. Suppose $\boldsymbol{u}$ is a two-dimensional incompressible, irrotational flow. Then there exists a velocity potential ϕ and stream function ψ such that

$$\begin{aligned} u &= \partial_x\phi = \partial_y\psi \\ v &= \partial_y\phi = -\partial_x\psi \end{aligned}$$

These constitute the Cauchy-Riemann (CR) equation. To this end, observe that ψ is also harmonic since

$$\boldsymbol{\omega} = (\partial_x v - \partial_y u)\,\boldsymbol{e}_3 = \partial_x(-\partial_x\psi) - \partial_y(\partial_y\psi) = -\Delta\psi\boldsymbol{e}_3 = 0.$$

5.1.1 Complex analysis

A complex function $f(z) = f(x + iy)$ is analytic if the following limit exists

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$

If f is analytic, then $\text{Re}(f)$ and $\text{Im}(f)$ satisfy the CR-equation. Conversely, if two functions $g(x, y)$ and $h(x, y)$ satisfy the CR-equation and their partial derivatives are continuous, then the function $f = g + ih$ is analytic. In our case, if ϕ, ψ are C^1 functions, then there exists a **complex potential** $w(z)$ such that $w(z) = \phi + i\psi$. How does the derivative of w relates to the fluid velocity field? Choose a path along the real-axis. Then

$$\begin{aligned} \frac{dw}{dz} &= \lim_{\Delta z \to 0} \frac{w(z + \Delta z) - w(z)}{\Delta z} \\ &= \lim_{\Delta x \to 0} \frac{\phi(x + \Delta x, y) + i\psi(x + \Delta x, y) - \phi(x, y) - i\psi(x, y)}{\Delta x} \\ &= \partial_x \phi + i\partial_x \psi \\ &= u - iv. \end{aligned}$$

We recover the same relation if we choose a path along the imaginary-axis, in which $\Delta z = i\Delta y$. A useful quantity is the magnitude of $w'(z)$ that is the flow speed:

$$q = \left| \frac{dw}{dz} \right| = |u^2 + v^2| = |\boldsymbol{u}|.$$

Example 5.1.8. Consider the uniform flow at an angle α extended from the x-axis, defined by

$$u = U \cos \alpha \quad \text{and} \quad v = U \sin \alpha$$

for some constant $U > 0$. Then

$$\frac{dw}{dz} = u - iv = U \cos \alpha - iU \sin \alpha = Ue^{-i\alpha}.$$

Since w is independent of z, we may integrate once and find

$$w = Ue^{-i\alpha} z = (U \cos \alpha - iU \sin \alpha)(x + iy) = \phi + i\psi$$

which implies that

$$\begin{aligned} \phi &= U \cos \alpha x + U \sin \alpha y \\ \psi &= -U \sin \alpha x + U \cos \alpha y. \end{aligned}$$

For $\alpha = 0$, $\phi = Ux$ and $\psi = Uy$.

Example 5.1.9. Consider the line vortex

$$\boldsymbol{u} = \frac{\Gamma}{2\pi} \frac{1}{r} \boldsymbol{e}_\theta.$$

One can show that Γ is the circulation. We have seen that $\phi = \dfrac{\Gamma}{2\pi}\theta$. The stream function satisfies

$$u_r = \frac{1}{r} \partial\theta \psi \quad \text{and} \quad u_\theta = -\partial_r \psi,$$

or

$$\partial_\theta \psi = 0 \quad \text{and} \quad -\partial_r \psi = \frac{\Gamma}{2\pi r}.$$

Integrating once, we find that

$$\psi = -\frac{\Gamma}{2\pi} \ln r.$$

Consequently,

$$w = \frac{\Gamma}{2\pi}[\theta - i \ln r] = -\frac{i\Gamma}{2\pi}[\ln r + i\theta] = -\frac{i\Gamma}{2\pi} \ln z$$

if we choose the principal branch of the complex logarithmic.

5.1.2 2/8/2018

2D incompressible irrotational flow past a rigid body. Body to be a streamline of the flow. We don't impose no-slip on the body.

Method of solutions:

1. Potential and stream functins
2. Method of images - Milne Thomson Circle theorem

Because the PDEs are linear, we can use superposition principle. Three steps

1. uniform flow past a cylinder
2. uniform flow past an elliptical cylinder
3. uniform flow past an aerofoil

Example 5.1.10. 1. Uniform flow at an angle α. The complex potential is

$$w = Uze^{-i\alpha}.$$

2. Line vortex at z_0. The complex potential is

$$w(z) = -\frac{i\Gamma}{2\pi} \ln(z - z_0),$$

where Γ is the circulation around $z = z_0$:

$$\Gamma = \oint_{C(z_0)} \boldsymbol{u} \cdot \boldsymbol{\gamma}.$$

Γ is also called the strength. For the line vortex, $\boldsymbol{u} = \frac{\Gamma}{2\pi r}\boldsymbol{e}_\theta$ and so the streamlines are circles.

3. Source/sink. $\phi = c \ln r, c \in \mathbb{R}$ and $\psi = c\theta$. $w(z) = c \ln z$ and the streamlines are $\theta = \text{constant}$.

4. Dipole/Doublet flow. $w(z) = A/z, A \in \mathbb{R}$.

$$\phi(x,y) = \frac{Ax}{x^2+y^2}, \quad \psi(x,y) = -\frac{Ay}{x^2+y^2}.$$

Streamlines are $\psi =$ constant, *i.e.* shifted circles along the y-axis. One can show that

$$u_r = -\frac{A}{r^2}\cos\theta, \quad u_\theta = -\frac{A}{r^2}\sin\theta.$$

Let us consider the problem of finding the flow generated by a vortex a distance from the origin on the x-axis in the half-space $x \geq 0$. Consider the free space and another line vortex of strength $-\Gamma$ at $x = -d$. By superposition,

$$\begin{aligned} w(z) &= w_+(z) + w_-(z) \\ &= -\frac{i\Gamma}{2\pi}\ln(z-d) + \frac{i\Gamma}{2\pi}\ln(z+d) \\ &= -\frac{i\Gamma}{2\pi}\ln\left(\frac{z-d}{z+d}\right) \\ &= -\frac{i\Gamma}{2\pi}\left[\ln\left|\frac{z-d}{z+d}\right| + i\arg\left(\frac{z-d}{z+d}\right)\right] \\ &= \phi(r,\theta) + i\psi(r,\theta). \end{aligned}$$

Consequently, the stream function is $\psi(r,\theta) = -\frac{\Gamma}{2\pi}\ln\left|\frac{z-d}{z+d}\right|$. The streamlines are $\left|\frac{z-d}{z+d}\right| = c$, *i.e.* these are circles. On the imaginary axis, observe that $\psi = 0$. One can check that the normal velocity component $u = 0$ on the boundary (the wall). What if the boundary is curved?

Theorem 5.1.11 (Milne-Thompson's Circle Theorem). *Let $f(z)$ be a complex function. Suppose that all singularities of $f(z)$ lie in $|z| > a$. Then the function*

$$w(z) = f(z) + \overline{f\left(\frac{a^2}{\bar{z}}\right)}$$

is the complex potential of a flow with

1. *the same singularities as f in $|z| > a$*
2. *$|z| = a$ is a streamline of $w(z)$.*

Proof. The singularities of $f\left(\frac{a^2}{\bar{z}}\right)$ are in $\left|\frac{a^2}{\bar{z}}\right| > a$, *i.e.* in $|z| < a$. On the circle $|z| = a$, observe that

$$\frac{a^2}{\bar{z}} = \frac{a^2 z}{|z|^2} = \frac{a^2 z}{a^2} = z.$$

This means that the complex function $w(z)$ restricted to the circle $|z| = a$ is

$$w\Big|_{|z|=a} = f(z) + \overline{f(z)} = 2\text{Re}\left(f(z)\right).$$

In particular, the imaginary part which is the streamline is zero at the circle $|z| = a$. ■

Let us consider the uniform flow past a cylinder with radius $a > 0$. In the absence of the cylinder, the flow is uniform and the complex potential is $f(z) = Uz$ and the only singularity is at $z = \infty$ (essential)? By the circle theorem,

$$\begin{aligned} w(z) &= f(z) + \overline{f\left(\frac{a^2}{\bar{z}}\right)} \\ &= Uz + \frac{Ua^2}{z} \end{aligned}$$

has the same singularities outside the circle and $|z| = a$ is a streamline. We claim that the circulation of the corresponding flow around the cylinder is 0. Let's work in polar coordinates.

$$\begin{aligned} w(r,\theta) &= Ure^{i\theta} + \frac{Ua^2}{r}e^{-i\theta} \\ &= U\left(r\cos\theta + ri\sin\theta + \frac{a^2}{r}\cos\theta - \frac{a^2}{r}i\sin\theta\right) \\ &= \phi(r,\theta) + i\psi(r,\theta) \end{aligned}$$

which gives $\phi(r,\theta) = U\cos\theta\left(r + \frac{a^2}{r}\right)$ and $\psi(r,\theta) = U\sin\theta\left(r - \frac{a^2}{r}\right)$. The velocity components are

$$\begin{aligned} u_r &= \frac{1}{r}\partial_\theta\psi = \partial_r\phi = U\cos\theta\left(1 - \frac{a^2}{r^2}\right) \\ u_\theta &= \partial_r\psi = \frac{1}{r}\partial_\theta\phi = -U\sin\theta\left(1 + \frac{a^2}{r^2}\right) \end{aligned}$$

The circulation around a circle $\gamma = R\boldsymbol{e}_r$ is

$$\begin{aligned} \Gamma = \oint_C \boldsymbol{u}\cdot\boldsymbol{\gamma} &= \int_0^{2\pi} \boldsymbol{u}(\gamma)\cdot\dot{\gamma}\,d\theta \\ &= \int_0^{2\pi} (u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta)\bigg|_{r=R} \cdot R\boldsymbol{e}_\theta\,d\theta \\ &= \int_0^{2\pi} u_\theta\bigg|_{r=R} R\,d\theta \\ &= -U\left(1 + \frac{a^2}{r^2}\right)R\int_0^{2\pi}\sin\theta\,d\theta = 0 \end{aligned}$$

Let us discuss about boundary conditions on the circle $|z| = a$. We know by construction that $\psi = 0$ on the circle. Moreover

$$\phi\bigg|_{r=a} = 2U\cos\theta, \quad u_r\bigg|_{r=a} = 0, \quad u_\theta\bigg|_{r=a} = -2U\sin\theta \neq 0.$$

The second one is no penetration, and the last one is the slip condition. There are two stagnation points at $\theta = 0, \pi$ in which $u_\theta = 0$ on the circle. Otherwise, there is slip on the body.

Now, let's impose the additional constraint that we want nonzero circulation around the cylinder. By superposition principle, we add a vortex of strength Γ at the origin. The corresponding complex potential is

$$w(z) = U\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln z.$$

Computing $w'(z)$ gives

$$w'(z) = U\left(1 - \frac{a^2}{z^2}\right) - \frac{i\Gamma}{2\pi}\frac{1}{z}$$

and as $|z| \longrightarrow \infty$, $w'(z) \longrightarrow U$. Let us find the velocity potential and stream function.

$$w(r,\theta) = U\left(r\cos\theta + ir\sin\theta + \frac{a^2}{r}\cos\theta - i\frac{a^2}{r}\sin\theta\right) - \frac{i\Gamma}{2\pi}\ln r - \frac{i\Gamma}{2\pi}i\theta$$

which implies

$$\phi(r,\theta) = U\cos\theta\left(r + \frac{a^2}{r^2}\right) + \frac{\Gamma}{2\pi}\theta$$
$$\psi(r,\theta) = -U\sin\theta\left(r - \frac{a^2}{r}\right) - \frac{\Gamma}{2\pi}\ln r.$$

The cylinder remains a streamline since

$$\psi\Big|_{r=a} = -\frac{\Gamma}{2\pi}\ln a.$$

To obtain $\psi = 0$ on the cylinder, we substract the corresponding constant from the contribution of stream function in $w(z)$

$$\begin{aligned} w(z) &= U\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln z + \frac{i\Gamma}{2\pi}\ln a \\ &= U\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln\left(\frac{z}{a}\right). \end{aligned}$$

Let $z = re^{i\theta}$ and define the dimensionless variable $\tilde{r} = r/a$ and $\tilde{z}$: $z/a = \tilde{r}e^{i\theta}$. Then

$$\begin{aligned} w(\tilde{z}) &= U\left(re^{i\theta} + \frac{a^2}{r}e^{-i\theta}\right) - \frac{i\Gamma}{2\pi}\ln\left(\frac{r}{a}e^{i\theta}\right) \\ &= Ua\left(\tilde{r}e^{i\theta} + \frac{1}{\tilde{r}}e^{-i\theta}\right) - \frac{i\Gamma}{2\pi}\ln\left(\tilde{r}e^{i\theta}\right) \\ &= Ua\left[\left(\tilde{r}e^{i\theta} + \frac{1}{\tilde{r}}e^{-i\theta} - \frac{i\Gamma}{2\pi Ua}\ln\left(\tilde{r}e^{i\theta}\right)\right)\right] \\ &= Ua\left(\tilde{z} + \frac{1}{\tilde{z}} - iB\ln\tilde{z}\right) \end{aligned}$$

Consider

$$\tilde{w}(\tilde{z}) = \tilde{z} + \frac{1}{\tilde{z}} - iB\ln\tilde{z}$$

$$= \tilde{r}\cos\theta + i\tilde{r}\sin\theta + \frac{1}{\tilde{r}}\cos\theta - \frac{i}{\tilde{r}}\sin\theta - iB\left(\ln\tilde{r} + i\theta\right).$$

We have that

$$\tilde{\psi}(\tilde{r},\theta) = \left(\tilde{r} - \frac{1}{\tilde{r}}\right)\sin\theta - B\ln\tilde{r}$$
$$\tilde{\phi}(\tilde{r},\theta) = \left(\tilde{r} + \frac{1}{\tilde{r}}\right)\cos\theta + B\theta.$$

Let us look at the stagnation point:

$$\frac{d\tilde{w}}{d\tilde{z}} = 1 - \frac{1}{\tilde{z}^2} - iB\frac{1}{\tilde{z}} = 0.$$

If $B < 2$, then the stagnation points are at $\tilde{r} = 1$ and $\sin\theta = -B/2$. If $B > 2$, then the stagnation points are $\tilde{r} = \frac{B}{2} + \sqrt{\frac{B^2}{4} - 1}$ and $\theta = \frac{3\pi}{2}$.

5.2 Lift

We claim that there is lift if $\Gamma \neq 0$, *i.e.* the force perpendicular to the U direction, $F_y \neq 0$. The lift is $-\rho U\Gamma$.

Theorem 5.2.1 (Kutta-Joukowski lift theorem)**.** *Consider the steady, irrotational ,incompressible flow past a 2D rigid body, the cross section of which is bounded by a simple curve C. Let the flow be uniform at infinity with speed U in the x-direction and Γ the circulation around the body. Then*

$$F_x = 0 \quad \text{and } F_y = -\rho U\Gamma$$

where $\boldsymbol{F} = (F_x, F_y)$ is the force on the body.

Theorem 5.2.2 (Blasius)**.** *If F_x, F_y are the components of the net force on the body, then*

$$F_x - iF_y = \frac{\rho i}{2}\oint_C \left(\frac{dw}{dz}\right)^2 dz,$$

where $w(z)$ is the complex potential.

Proof. Recall the (steady) Bernoulli's principle with no external force

$$\frac{1}{2}|\boldsymbol{u}^2| + \frac{p}{\rho} = C.$$

Choosing $C = p_0/\rho$, then

$$p = p_0 - \frac{\rho}{2}|\boldsymbol{u}|^2.$$

Note that

$$|\boldsymbol{u}|^2 = \left|\frac{dw}{dz}\right|^2 = \frac{dw}{dz}\overline{\frac{dw}{dz}} = \frac{dw}{dz}\frac{d\bar{w}}{d\bar{z}}.$$

We claim that

$$F_x = \oint_C -p\,dy \quad \text{and} \quad F_y = \oint_C p\,dx.$$

Assuming this,

$$\begin{aligned} F_x - iF_y &= \frac{1}{2}\rho \oint_C \left|\frac{dw}{dz}\right|^2 (dy + idx) \\ &= \frac{\rho i}{2} \oint_C \left|\frac{dw}{dz}\right|^2 (-idy + dx) \\ &= \frac{\rho i}{2} \oint_C \frac{dw}{dz}\frac{d\bar{w}}{d\bar{z}}\, d\bar{z} \end{aligned}$$

■

For the cylinder with lift,

$$w(z) = U\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln(z)$$

and

$$\begin{aligned} \frac{dw}{dz} &= U\left(1 - \frac{a^2}{z^2}\right) - \frac{i\Gamma}{2\pi z} \\ \left(\frac{dw}{dz}\right) &= U^2\left(1 - \frac{a^2}{z^2}\right)^2 - \frac{\Gamma^2}{4\pi^2 z^2} - \frac{2i\Gamma U}{2\pi z}\left(1 - \frac{a^2}{z^2}\right) \\ &= U^2 - \frac{2U^2a^2}{z^2} + \frac{U^2a^4}{z^4} - \frac{\Gamma^2}{4\pi^2z^2} - \frac{i\Gamma U}{\pi z} - \frac{i\Gamma U a^2}{z^2}. \end{aligned}$$

Since

$$\oint_C z^n\, dz = \begin{cases} 0 & \text{if } n \neq -1, \\ 2\pi i & \text{if } n = 1. \end{cases}$$

where C is a circle of radius a, then

$$F_x - iF_y = \frac{1}{2}\rho i\left[-\frac{i\Gamma U}{\pi} 2\pi i\right] = \rho i \Gamma U.$$

Note that $F_y = -\rho\Gamma U > 0$ if $\Gamma < 0$.

5.3 Conformal map

Definition 5.3.1. A Mobius transform is the transformation

$$S(z) = \frac{az + b}{cz + d} \quad \text{with } ad - bc \neq 0$$

Define the **Joukowsky transformation** as the following conformal mapping:

$$J(z) = \xi = z + \frac{c^2}{z}, \quad c \in \mathbb{R}.$$

Facts about Joukowsky transformation:

1. The inverse transform is
$$z = \frac{1}{2}\xi + \left(\frac{\xi^2}{4} - c^2\right)^{1/2}.$$
The branch points are at $\pm 2c$. We pick the branch cut connecting $z = -2c$ and $z = 2c$.

2. $J'(\pm c) = 0,\ J''(\pm c) \neq 0$.

Example 5.3.2. Find the image of $z = ae^{i\theta}, 0 \leq c < a$ under $J(z)$. This is an ellipse.

5.4 Thermal Instabilities

5.4.1 Benard convection

We begin with the Boussinesq's approximation. Because the temperature is not constant, so is the density. As a result, both the density and pressure are now functions of temperature, *i.e.* $\rho = \rho(T)$ and $p = p(T)$. Recall the system of conservation equations for a Newtonian fluid with non-constant $\mu = \mu(\boldsymbol{x})$:

$$\rho\Big(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\Big) = -\nabla p + \rho\boldsymbol{g} + \mu\Delta\boldsymbol{u} + \nabla\mu\cdot\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right)$$
$$\frac{1}{\rho}\Big(\partial_t\rho + \boldsymbol{u}\cdot\nabla\rho\Big) + \nabla\cdot\boldsymbol{u} = 0$$

The temperature equation takes the form

$$\rho C_p\Big(\partial_t T + \boldsymbol{u}\cdot\nabla T\Big) = \nabla\cdot(k\nabla T)\,, \quad k = k(\boldsymbol{x}).$$

where the second and third terms correspond to convection and diffusion. The distinction with the usual case is that ρ, μ, k, C_p are now functions of T. Let $\bar{\rho}, \bar{\mu}, \bar{k}, \overline{C_p}, \bar{T}$ be the ambient constants. The Boussinesq approximation is as follows. Suppose $\Delta T = T - \bar{T}$ is "small" such that

$$\frac{\bar{\rho}}{\rho} = 1 + \alpha\Delta T$$
$$\frac{\bar{\mu}}{\mu} = 1 + \beta\Delta T$$
$$\frac{\bar{k}}{k} = 1 + \gamma\Delta T$$
$$\frac{\overline{C_p}}{C_p} = 1 + \delta\Delta T$$

with $\alpha\Delta T, \beta\Delta T, \gamma\Delta T, \delta\Delta T \ll 1$. We may approximate ρ, μ, k, C_p using geometric series. For instance,
$$\rho = \frac{\bar{\rho}}{1 + \alpha\Delta T} \approx \bar{\rho}\Big(1 - \alpha\Delta T\Big).$$
In the Boussinesq approximation, the conversation of mass equation yields the incompressibility condition $\nabla\cdot\boldsymbol{u} = 0$. The conservation of temperature equation yields

$$\bar{\rho}\overline{C_p}\Big(\partial_t T + \boldsymbol{u}\cdot\nabla T\Big) = \bar{k}\nabla^2 T.$$

We may define $\kappa = \dfrac{\bar{k}}{\bar{\rho}\bar{C}_p}$. Let $p = p_H + p_D$, where p_H is the hydrostatic pressure satisfying

$$0 = -\nabla p_H + \bar{\rho}\boldsymbol{g}.$$

Consequently,

$$\bar{\rho}(1 - \alpha\Delta T)\boldsymbol{g} - \Delta p = -\bar{\rho}\alpha\Delta T\boldsymbol{g} - \nabla p_D$$

and the conservation of linear momentum equation reduces to (again, assuming all the four quantities above are small)

$$\bar{\rho}\Big(\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\Big) = -\nabla p_D + \bar{\mu}\nabla^2\boldsymbol{u}.$$

This is too perfect, in the sense that we lose the assumption that the fluid is convected by the thermal gradient. The second part of the Boussinesq's approximation is to keep the buoyancy term $-\bar{\rho}\alpha\Delta T\boldsymbol{g}$. As a consequence,

$$\bar{\rho}\Big(\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\Big) = -\alpha\Delta T\bar{\rho}\boldsymbol{g} - \nabla p_D + \bar{\mu}\nabla^2\boldsymbol{u}.$$

All the reduced equations are still nonlinear, so we are going to linearise them around the base (ambient) state.

5.5 Vortex Dynamics

Theorem 5.5.1 (Kelvin's Circulation Theorem)**.** *Let an inviscid, incompressible fluid with constant density be in motion in the presence of a conservative body force $\boldsymbol{f} = -\nabla\chi$ per unit mass. Let $C(t)$ be a closed material curve consisting of the same fluid particles as time proceeds. The circulation*

$$\Gamma = \int_{C(t)} \boldsymbol{u}\cdot d\boldsymbol{x}$$

around $C(t)$ is independent of time, i.e. $\Gamma(t) \equiv \Gamma(0)$ for all $t > 0$.

Proof. We parameterise $C(t)$ as

$$C(t) = \{\boldsymbol{x}(s,t)\colon 0 \le s \le 1\}.$$

Then

$$\begin{aligned}
\frac{d}{dt}\Gamma(t) = \frac{d}{dt}\left(\int_{C(t)} \boldsymbol{u}(\boldsymbol{x},t)\,d\boldsymbol{x}\right) &= \frac{d}{dt}\left(\int_0^1 \boldsymbol{u}(\boldsymbol{x}(s,t),t)\cdot\frac{\partial\boldsymbol{x}(s,t)}{\partial s}\,ds\right) \\
&= \int_0^1 \frac{D\boldsymbol{u}}{Dt}\cdot\frac{\partial\boldsymbol{x}(s,t)}{\partial x}\,ds + \int_0^1 \boldsymbol{u}(\boldsymbol{x}(s,t),t)\cdot\frac{\partial\boldsymbol{u}(\boldsymbol{x}(s,t),t)}{\partial s}\,ds \\
&= \int_0^1 \frac{D\boldsymbol{u}}{Dt}\cdot\frac{\partial\boldsymbol{x}(s,t)}{\partial x}\,ds + \int_0^1 \frac{1}{2}\frac{\partial}{\partial s}\left\{\boldsymbol{u}(\boldsymbol{x}(s,t),t)\cdot\boldsymbol{u}(\boldsymbol{x}(s,t),t)\right\}\,ds.
\end{aligned}$$

The second integral vanishes since $C(t)$ is a closed curve. Rewriting the material derivative of $\boldsymbol{u}$ using Euler equation, we see that

$$\frac{d}{dt}\Gamma(t) = \int_0^1 -\nabla\left(\frac{p}{\rho} + \chi\right)\cdot\frac{\partial\boldsymbol{x}(s,t)}{\partial s}\,ds$$

$$= \int_{C(t)} -\nabla \left(\frac{p}{\rho} + \chi \right) \cdot d\boldsymbol{x} = 0,$$

since we are integrating a conservative vector field over a closed curve.

■

Remark 5.5.2. The theorem does not require the fluid region to be simply connected. The Euler equation enters the proof in evaluating the line integral around C, so the theorem continues to hold even if the viscous effect happens to be important elsewhere in the flow.

Definition 5.5.3. A **vortex line** is a curve which is everywhere tangent to the vorticity $\boldsymbol{\omega}(\boldsymbol{x}, t)$. The vortex line which pass through some simple closed curve in space are said to form the boundary of a **vortex tube**.

Theorem 5.5.4 (Helmholtz)**.** *Suppose an inviscid, incompressible fluid moves in the pressure of body force $\boldsymbol{f} = -\nabla\chi$ per unit mass. Then*

1. *The fluid particles which lie on a vortex line at some time continue to lie on a vortex line as time advances, i.e. vortex lines move with the fluid. An immediate consequence is that vortex tubes move with the fluid.*

2. *The quantity*

$$\Gamma = \int_S \boldsymbol{\omega} \cdot \boldsymbol{n}\, dS$$

*is the same for all cross-sections S of a vortex tube and is also independent of time. Γ is called the **strength** of the vortex tube.*

Proof. Let us define a *vortex surface* as a surface such that $\boldsymbol{\omega}$ is everywhere tangent to the surface. Suppose that at $t = 0$, the vortext line is the intersection of two vortex surfaces S_1 and S_2. On each of these surfaces,

$$\boldsymbol{\omega}(\boldsymbol{x}, 0) \cdot \boldsymbol{n}_i = 0 \quad \text{where } \boldsymbol{n}_i \text{ is normal to } S_i, \text{ i=1,2.}$$

We claim that both S_1, S_2 remain as vortex surfaces as time advances, *i.e.* as the points on S_i move with the fluid, the surface they comprise at any time t is a vortex surface. Choose any arbitrary closed material curve $C_1(t)$ that encloses a surface $S_1^*(t) \subset S_1(t)$. It follows from Stokes theorem that

$$\int_{C_1(0)} \boldsymbol{u} \cdot d\boldsymbol{x} = \int_{S_1^*(0)} \boldsymbol{\omega} \cdot \boldsymbol{n}\, dS = 0.$$

This quantity remains zero by virtue of Kelvin's circulation theorem. Since $C_1(t)$ and so $S_1^*(t)$ was arbitrary, this shows that $\boldsymbol{\omega} \cdot \boldsymbol{n}_1 = 0$ on $S_1(t)$ and an identical argument shows that $\boldsymbol{\omega} \cdot \boldsymbol{n}_2 = 0$ on $S_2(t)$.

We proceed to prove the second statement. Consider any two arbitrary cross sections A_1, A_2 of a vortex tube and consider the vortex tube bounded by these two cross sections. Since $\boldsymbol{u}$ is incompressible,

$$\nabla \cdot \boldsymbol{\omega} = \nabla \cdot (\nabla \cdot \boldsymbol{u}) = 0$$

and it follows from the divergence theorem that

$$0 = \int_V \nabla \cdot \boldsymbol{\omega}\, d\boldsymbol{x} = \int_{\partial V} \boldsymbol{\omega} \cdot \boldsymbol{n}\, dS = \int_{A_1 \cup A_2} \boldsymbol{\omega} \cdot \boldsymbol{n}\, dS.$$

Let $C(t)$ be a closed material curve which lie on the wall of the vortex tube, enclosing a closed region $A(t)$. Then

$$\int_{A(t)} \boldsymbol{\omega}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x},t)\,dS = \int_{C(t)} \boldsymbol{u}(\boldsymbol{x},t)\cdot\,d\boldsymbol{x}.$$

It follows from Kelvin's circulation theorem that Γ remains constant as time proceeds.

■

Another statement of Helmholtz is that a parcel of fluid that is initially irrotationaly remains irrotational at later times. Recall the vorticity equation

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} + \nu\Delta\boldsymbol{\omega}.$$

If $\boldsymbol{\omega}(\boldsymbol{x},0)\equiv 0$, then $\boldsymbol{\omega}(\boldsymbol{x},t)\equiv 0$ for all $t>0$. In particular, the vorticity equation doesn't tell us how the vorticity is generated. At this point, it seems conceivable that the vorticity can be generated at no-slip boundary. To this end, consider the Rayleigh problem (impulsively moved plane with speed U). We found a similarity solution

$$u(y,t) = U\left[1 - \frac{1}{\sqrt{\pi\nu}}\int_0^\eta e^{-s^2/4}\,ds\right], \quad \text{where } \eta = y/\sqrt{\nu t}.$$

The vorticity is

$$\boldsymbol{\omega}(y,t) = -\partial_y u(y,t) = \frac{U}{\sqrt{\pi\nu t}}e^{-y^2/4\nu t}.$$

See Acheson, page 37 and 38 for a discussion of this.

Now, suppose at $t=0$ we have an aerofoil that moves to the left with speed U. By virtue of the vorticity equation and Kelvin's circulation theorem, the circulation around the aerofoil remains zero for later time. This implies that to generate lift..

5.5.1 Helmholtz Decomposition

Given the vorticity $\boldsymbol{\omega}$, can we find a unique velocity field $\boldsymbol{u}$ such that $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$? As stated, if $\boldsymbol{u}_1$ is a solution to this problem, then $\boldsymbol{u}_1 + \nabla\phi$ is also a solution for any scalar function ϕ, since

$$\boldsymbol{\omega} = \nabla\times(\boldsymbol{u}_1 + \nabla\phi) = \nabla\times\boldsymbol{u}_1 + \nabla\times\nabla\phi = \nabla\times\boldsymbol{u}_1.$$

So the solution is unique up to addition of potential flow. Suppose $\boldsymbol{\omega}(\boldsymbol{x})$ is smooth and decays sufficiently fast as $|\boldsymbol{x}|\longrightarrow\infty$. Since the voriticity equation was derived under the assumption that $\boldsymbol{u}$ is incompressible, we modify our problem: Given $\boldsymbol{\omega}$, find $\boldsymbol{u}$ such that $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$ and $\nabla\cdot\boldsymbol{u} = 0$. Again, suppose $\boldsymbol{u}_1$ is a solution to this problem and consider $\boldsymbol{u}_2 = \boldsymbol{u}_1 + \nabla\phi$. It follows that for $\boldsymbol{u}_2$ to be a solution again, we need $\Delta\phi = 0$. If ϕ is bounded, then it is constant so that $\nabla\phi = 0$. Hence the solution to this modified problem, if it exists, must be unique. Suppose we have such a $\boldsymbol{u}$, then

$$\begin{aligned}\nabla\times\boldsymbol{\omega} &= \nabla\times(\nabla\times\boldsymbol{u})\\ &= \nabla(\nabla\times\boldsymbol{u}) - \Delta\boldsymbol{u}\\ &= -\Delta\boldsymbol{u}.\end{aligned}$$

This is the standard Poisson problem with data $\nabla \times \boldsymbol{\omega}$, with solution

$$\boldsymbol{u} = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\nabla_{\boldsymbol{y}} \times \boldsymbol{\omega}(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|} \, d\boldsymbol{y}.$$

Let us manipulate this. Looking at the ith component,

$$\begin{aligned}
u_i(\boldsymbol{y}) &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\varepsilon_{ijk} \partial_{y_j} \omega_k}{|\boldsymbol{x} - \boldsymbol{y}|} \, d\boldsymbol{y} \\
&= \frac{1}{4\pi} \varepsilon_{ijk} \int_{\mathbb{R}^3} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} \frac{\partial \omega_k}{\partial y_j} \, d\boldsymbol{y} \\
&= -\frac{1}{4\pi} \varepsilon_{ijk} \int_{\mathbb{R}^3} \frac{\partial}{\partial y_j} \left(\frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} \right) \omega_k \, d\boldsymbol{y} \\
&= -\frac{1}{4\pi} \varepsilon_{ijk} \int_{\mathbb{R}^3} \frac{x_j - y_j}{|\boldsymbol{x} - \boldsymbol{y}|^3} \omega_k \, d\boldsymbol{y} \\
&= -\frac{1}{4\pi} \int_{\mathbb{R}^3} \varepsilon_{ijk} \frac{x_j - y_j}{|\boldsymbol{x} - \boldsymbol{y}|^3} \omega_j \, d\boldsymbol{y} \\
&= -\frac{1}{4\pi} \left[\int_{\mathbb{R}^3} \left(\frac{\boldsymbol{x} - \boldsymbol{y}}{|\boldsymbol{x} - \boldsymbol{y}|^3} \right) \times \boldsymbol{\omega}(\boldsymbol{y}) \, d\boldsymbol{y} \right]_i \\
&= \frac{1}{4\pi} \left[\int_{\mathbb{R}^3} \frac{\boldsymbol{\omega}(\boldsymbol{y}) \times (\boldsymbol{x} - \boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^3} \, d\boldsymbol{y} \right]_i .
\end{aligned}$$

Consequently,

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\boldsymbol{\omega}(\boldsymbol{y}) \times (\boldsymbol{x} - \boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^3} \, d\boldsymbol{y}.$$

This is somewhat related/similar to the Biot-Savart Law.

Consider an axisymmetric situation. In cylindrical coordinates,

$$\boldsymbol{u}(r, z) = u_r(r, z) \boldsymbol{e}_r + u_z(r, z) \boldsymbol{e}_z.$$

One can show that

$$\boldsymbol{\omega}(r, z) = \omega(r, z) \boldsymbol{e}_\theta.$$

Chapter 6

Stokes Flow

We will study viscous dominated flow, or so called creeping. Recall the incompressible Navier-Stokes equation:

$$\rho\left(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \mu\Delta\boldsymbol{u}$$
$$\nabla\cdot\boldsymbol{u} = 0.$$

Choosing the characteristic length ℓ_c and velocity $\boldsymbol{u}_c$, we have two options for the characteristics time t_c:

1. choose t_c such that $\boldsymbol{u}_c = \ell_c/t_c$;
2. choose t_c such that $\boldsymbol{u}_c \neq \ell_c/t_c$.

The dimensionless momentum equation is

$$\rho\left(\frac{u_c}{t_c}\partial_{t'}\boldsymbol{u}' + \frac{u_c^2}{\ell_c}\boldsymbol{u}'\cdot\nabla\boldsymbol{u}'\right) = -p_c\nabla p' + \mu\left(\frac{u_c}{\ell_c^2}\right)\Delta\boldsymbol{u}'$$
$$\left(\frac{\rho\ell_c^2}{\mu t_c}\right)\partial_{t'}\boldsymbol{u}' + \left(\frac{\rho u_c\ell_c}{\mu}\right)\boldsymbol{u}'\cdot\nabla\boldsymbol{u}' = -\left(\frac{p_c\ell_c^2}{\mu u_c}\right)\nabla p' + \Delta\boldsymbol{u}'$$

Define the dimensionless parameter

$$\mathrm{Re} = \frac{\rho u_c\ell_c}{\mu} = \text{Reynolds number}$$
$$\mathrm{St} = \frac{u_c t_c}{\ell_c} = \text{Strouhal (Stokes) number.}$$

Choosing $p_c = \dfrac{\mu u_c}{\ell^2}$, the dimensionless momentum equation reduces to

$$\mathrm{Re}\left(\frac{1}{\mathrm{St}}\partial_{t'}\boldsymbol{u}' + \boldsymbol{u}'\cdot\nabla\boldsymbol{u}'\right) = \Delta\boldsymbol{u}' - \nabla p'.$$

For $\mathrm{Re} \ll 1$ and $\mathrm{St} = 1$ or $\mathrm{Re} \ll 1$ and $\dfrac{\mathrm{Re}}{\mathrm{St}} \ll 1$, we obtain the Stokes equation

$$0 = -\nabla p + \Delta\boldsymbol{u}$$
$$\nabla\cdot\boldsymbol{u} = 0.$$

For Re $\ll 1$ only, we obtain the unsteady Stokes equation

$$\left(\frac{\text{Re}}{\text{St}}\right)\partial_t \boldsymbol{u} = -\nabla p + \Delta \boldsymbol{u}$$
$$\nabla \cdot \boldsymbol{u} = 0.$$

For the Stokes equation, the time factor only enters through the boundary conditions. One important consequence is that Stokes flow is reversible (in time and in space)! Also, in Stokes flow, the sphere stays at a constant distance from the wall.

Theorem 6.0.1 (Lorentz Reciprocal Identity)**.** *Let* $\left(\boldsymbol{u}_1, \underline{\underline{\sigma}}_1\right)$, $\left(\boldsymbol{u}_2, \underline{\underline{sigma}}_2\right)$ *be two solutions of Stokes flow outside a body with boundary* S*. Assume that* $\boldsymbol{u}_1, \boldsymbol{u}_2 \longrightarrow 0$ *as* $|\boldsymbol{x}| \longrightarrow \infty$ *and* $\boldsymbol{u}_1 = \boldsymbol{u}_2 = U$ *on* S*. Then*

$$\int_S \boldsymbol{u}_1 \cdot \underline{\underline{\sigma}}_2 \boldsymbol{n}\, dS = \int_S \boldsymbol{u}_2 \cdot \underline{\underline{\sigma}}_1 \boldsymbol{n}\, dS$$

or

$$\nabla \cdot \left(\boldsymbol{u}_1 \underline{\underline{\sigma}}_2 - \boldsymbol{u}_2 \underline{\underline{\sigma}}_1\right) = 0.$$

Proof. From product rule,

$$\begin{aligned}
u_{2,i}\partial_j\sigma_{1,ij} &= \partial_j\left(u_{2,i}\sigma_{1,ij}\right) - \left(\partial_j u_{2,i}\right)\sigma_{1,ij} \\
&= \partial_j\left(u_{2,i}\sigma_{1,ij}\right) - \left(\partial_j u_{2,i}\right)\left[-p_1\delta_{ij} + \mu\left(\partial_j u_{1,i} + \partial_i u_{1,j}\right)\right] \\
&= \partial_j\left(u_{2,i}\sigma_{1,ij}\right) - \mu\partial_j u_{2,i}\left(\partial_j u_{1,i} + \partial_i u_{1,j}\right) + p_1\partial_j u_{2,j} \\
&= \partial_j\left(u_{2,i}\sigma_{1,ij}\right) - \mu\partial_j u_{2,i}\left(\partial_j u_{1,i} + \partial_i u_{1,j}\right)
\end{aligned}$$

where the last term vanishes due to incompressibility of $\boldsymbol{u}_2$. A symmetric argument shows that

$$u_{1,i}\partial_j\sigma_{2,ij} = \partial_j\left(u_{1,i}\sigma_{2,ij}\right) - \mu\partial_j u_{1,i}\left(\partial_j u_{2,i} + \partial_i u_{2,j}\right). \tag{6.0.1}$$

Substracting these two equations results in

$$\begin{aligned}
&u_{2,i}\partial_j\sigma_{1,ij} - u_{1,i}\partial_j\sigma_{2,ij} \\
&= \partial_j\left(u_{2,i}\sigma_{1,ij} - u_{1,i}\sigma_{2,ij}\right) - \mu\left(\partial_j u_{1,i}\partial_j u_{1,i} + \partial_j u_{2,i}\partial_i u_{1,j} - \partial_j u_{1,i}\partial_j u_{2,i} - \partial_j u_{1,i}\partial_i u_{2,j}\right) \\
&= \partial_j\left(u_{2,i}\sigma_{1,ij} - u_{1,i}\sigma_{2,ij}\right).
\end{aligned}$$

In tensor notation, we have

$$\nabla \cdot \left(\boldsymbol{u}_2 \cdot \underline{\underline{\sigma}}_1 - \boldsymbol{u}_1 \cdot \underline{\underline{\sigma}}_2\right) = \boldsymbol{u}_2 \nabla \cdot \underline{\underline{\sigma}}_1 - \boldsymbol{u}_1 \nabla \cdot \underline{\underline{\sigma}}_2 = 0,$$

since $\nabla \cdot \underline{\underline{\sigma}}_j = 0, j = 1, 2$ is just the Stokes equation. The statement follows. ■

Theorem 6.0.2. *There exists a unique solution to Stokes flow in a volume* V *with boundary* S*.*

Proof. Assume that $\boldsymbol{v}_1, \boldsymbol{v}_2$ are two distinct solutions of Stokes equation. Linearity implies that $\boldsymbol{u} = \boldsymbol{v}_1 - \boldsymbol{v}_2$ is also a solution of Stokes equation and moreover $\boldsymbol{u} = \boldsymbol{0}$ on S.

$$\begin{aligned} 0 = \int_V \boldsymbol{u} \nabla \cdot \underline{\underline{\sigma}} \, dV &= \int_V \boldsymbol{u} \cdot (\nabla p - \mu \Delta \boldsymbol{u}) \ dV \\ &= \int_V \nabla \cdot (p\boldsymbol{u}) - p \nabla \cdot \boldsymbol{u} - \mu \partial_j (u_i \partial_j u_i) + \mu (\partial_j \boldsymbol{u}_i)^2 \ dV \\ &= \int_S [u_j p - \mu u_i \partial_j u_i] \, n_j \, dS + \mu \int_V (\partial_j v_i)^2 \ dV \\ &= \mu \int_V (\partial_j u_i)^2 \ dV \end{aligned}$$

This implies tht $\partial_j u_i = 0$ in V and so $\boldsymbol{u} = \boldsymbol{0}$ due to the boundary condition. ■

The simplest non-trivial solution of Stokes flow is $u = U$ and $p =$ constant. Taking the divergence of Stokes equation, we see that the pressure function is harmonic since

$$0 = -\nabla p + \Delta \nabla \cdot \boldsymbol{u} = -\nabla p.$$

Taking the curl of Stokes equation, we see that

$$0 = -\nabla \times \nabla p + \Delta \nabla \times \boldsymbol{u} = \Delta \boldsymbol{\omega}.$$

Last, if $\boldsymbol{u} = \nabla \phi$, then $\Delta^2 \phi = 0$. We are interested in solving Stokes flow past a sphere or axisymmetric bodies. We will solve the 2D Stokes flow using stream function, this also involves finding the fundamental solutions of Stokes flow. We point out the Stokes paradox: we cannot solve the Stokes flow past a disk (2D) or Stokes flow past an infinite-long cylinder.

6.1 Stream Function

A two-dimensional flow field $\boldsymbol{u} = (u, v, w)$ can be rewrite in terms of the stream function, as

$$\boldsymbol{u} = \nabla \psi \times \boldsymbol{e}_z = (\partial_y \psi, -\partial_x \psi, 0) .$$

One can easily check that $\boldsymbol{u}$ given by this is indeed incompressible. Recall the Stokes equation

$$0 = -\nabla p + \mu \Delta \boldsymbol{u}.$$

Taking the curl of this, one can show that

$$\Delta \omega = \Delta (-\Delta \psi) = -\Delta^2 \psi = 0,$$

where the vorticity is given by $\boldsymbol{\omega} = \omega \boldsymbol{e}_z$. We solve it in terms of a system:

$$\begin{aligned} \Delta \psi &= -\omega \\ \Delta \omega &= 0. \end{aligned}$$

Converting to polar coordinates, we have

$$\frac{1}{r} \partial_r (r \partial_r \psi) + \frac{1}{r^2} \partial_{\theta\theta} \psi = -\omega$$

$$\frac{1}{r}\partial_r\left(r\partial_r\omega\right) + \frac{1}{r^2}\partial_{\theta\theta}\omega = 0.$$

Using the method of separation of variables, let $\omega = R(r)F(\theta)$ and $\psi = S(r)H(\theta)$. The solutions are

$$\begin{aligned}
\omega &= (a_0 + b_0\theta)(c_0 + d_0\ln r) + \sum_{n=1}^{\infty}(a_n\cos(\lambda_n\theta) + b_n\sin(\lambda_n\theta))\left(c_n r^{-\lambda_n} + d_n r^{\lambda_n}\right)\\
\psi &= (a_0 + b_0\theta)\left(c_0 + d_0\ln r + \hat{c_0}r^2 + \hat{d_0}r^2\left(\ln r - \frac{1}{2}\right)\right)\\
&\quad + (a_1\sin\theta + b_1\cos\theta)\left(c_1 r + d_1 r^{-1} + \hat{c_1}r^3 + \hat{d_1}r\ln r\right)\\
&\quad + \sum_{n=2}^{\infty}(a_n\sin(\lambda_n\theta) + b_n\cos(\lambda_n\theta))\left(c_n r^{\lambda_n} + d_n r^{-\lambda_n} + \hat{c_n}r^{\lambda_n+2} + \hat{d_n}r^{2-\lambda_n}\right)
\end{aligned}$$

6.1.1 Corner flow

Consider the flow past a wedge. The boundary conditions are

$$\begin{aligned}
u_r = U,\quad u_\theta = 0 &\quad \text{at } \theta = 0\\
u_r = u_\theta = 0 &\quad \text{at } \theta = \alpha
\end{aligned}$$

In polar coordinates

$$u_r = \frac{1}{r}\partial_\theta\psi,\quad u_\theta = -\partial_r\psi.$$

At $\theta = 0$, we have

$$\frac{1}{r}S(r)H'(0) = U \quad \text{for all } r.$$

In particular, this implies that $\frac{S(r)}{r}$ must be constant and so we modify our ansatz for ψ as $\psi(r,\theta) = rH(\theta)$. Substituting into $\Delta\psi = \omega$ yields

$$\begin{aligned}
\partial_r(rH) + \frac{1}{r^2}(rH'') &= -R(r)F(\theta)\\
\frac{1}{r}(H + H'') &= -R(r)F(\theta)
\end{aligned}$$

which implies that $R(r) = -\dfrac{1}{r}$ and $F(\theta) = H(\theta) + H''(\theta)$, *i.e.* $\omega = -\frac{1}{r}(H + H'')$. Substituting this into $\Delta\omega = 0$ yields

$$\begin{aligned}
\frac{1}{r}\partial_r\left(r\frac{1}{r^2}(H + H'')\right) + \frac{1}{r^2}\left(-\frac{1}{r}\right)(H'' + H'''') &= 0\\
-\frac{1}{r^3}\Big[(H + H'') + (H + H'')''\Big] &= 0.
\end{aligned}$$

Let $\sigma(\theta) = H(\theta) + H''(\theta)$. Then the above reduces to

$$\sigma(\theta) + \sigma''(\theta) = 0$$

with general solutions of the form

$$\sigma(\theta) = a_1 \cos\theta + b_1 \sin\theta.$$

Consequently,

$$H(\theta) + H''(\theta) = a_1 \cos\theta + b_1 \sin\theta$$

and its general solution is

$$H(\theta) = A_1 \sin\theta + B_1 \cos\theta + C_1\theta \sin\theta + D_1\theta\cos\theta.$$

The boundary conditions for $H(\theta)$ are $H'(0) = U$, $H(0) = 0$, $H'(\alpha) = 0$ and $H(\alpha) = 0$:

$$\begin{aligned}
H'(0) &= A_1 + D_1 = U \\
H(0) &= B_1 = 0 \\
H'(\alpha) &= (A_1 + D_1)\cos\alpha + C_1(\sin\alpha + \alpha\cos\alpha) - D_1\alpha\sin\alpha \\
&= (A_1 + D_1)\cos\alpha + C_1\sin\alpha + \alpha(C_1\cos\alpha - D_1\sin\alpha) \\
&= U\cos\alpha + C_1\sin\alpha + \alpha(C_1\cos\alpha - D_1\sin\alpha) \\
H(\alpha) &= A_1\sin\alpha + C_1\alpha\sin\alpha + D_1\alpha\cos\alpha \\
&= (U - D_1)\sin\alpha + C_1\alpha\sin\alpha + D_1\alpha\cos\alpha = 0
\end{aligned}$$

The solution to this linear system is

$$\begin{aligned}
B_1 &= 0 \\
C_1 &= \frac{U(\alpha - \sin\alpha\cos\alpha)}{\sin^2\alpha - \alpha} \\
D_1 &= \frac{U\sin^2\alpha}{\sin^2\alpha - \alpha} \\
A_1 &= \frac{U\alpha^2}{\sin^2\alpha - \alpha}.
\end{aligned}$$

It can be shown that the shear stress along $\theta = 0$ is

$$\tau_{r\theta}\bigg|_{\theta=0} = -\left(\frac{2U\mu}{\sin^2\alpha - \alpha}\right)\frac{1}{r}(\sin\alpha\cos\alpha - \alpha)$$

and this has a singularity at $r = 0$.

6.1.2 3D axisymmetric

Define the differential operator

$$E^2 = \partial_{rr} + \left(\frac{1-\eta}{r^2}\right)\partial_{\eta\eta}, \quad \text{where } \eta = \cos\theta.$$

Then $E^4\psi = 0$, where $\psi = \psi(r, \eta, \phi)$, $-1 \le \eta \le 1$ and

$$u_r = -\frac{1}{r^2}\partial_\eta\psi \quad \text{and} \quad u_\theta = -\frac{1}{\sqrt{1-\eta^2}}\frac{1}{r}\partial_r\psi.$$

We are still solving it using separation of variables, but we need to use Gegenbauer polynomial Q_n defined by

$$Q_n(\eta) = \int_{-1}^{\eta} P_n(x)\, dx,$$

where $P_n(x)$ is Legendre polynomial. For example,

$$Q_1(\eta) = \frac{1}{2}\left(\eta^2 - 1\right), \quad Q_2(\eta) = \frac{1}{2}\left(\eta^3 - \eta\right).$$

We may recover the solution for the sphere and also obtain the solution for an ellipsoid.

6.1.3 Stokes stream function for 3D axisymmetric problem

Consider a curvilinear coordinate system (q_1, q_2, q_3) with scale factors h_1, h_2, h_3 such that

$$(ds)^2 = \frac{1}{h_1^2}(dq_1)^2 + \frac{1}{h_2^2}(dq_2)^2 + \frac{1}{h_3^2}(dq_3)^2 = \frac{1}{h_j}^{2}(dq_j)^2.$$

[This is the metric tensor, we are using Leal's notation here, wolfram alpha defines h_1, h_2, h_3 as the inverse of Leal]. For example, $h_1 = 1, h_2 = \frac{1}{r}, h_3 = \frac{1}{r\sin\theta}$ are the scale factors for spherical coordinates. Let

$$\boldsymbol{u} = (h_2 h_3 \partial_{q_2}\psi, -h_1 h_3 \partial_{q_1}\psi, 0)$$

where $\boldsymbol{e}_{q_3}$ is the axis of symmetry. One can show that

1. $\boldsymbol{u} \cdot \nabla\psi = 0$.
2. ψ is constant on the stream tube.
3. ψ satisfies $E^4\psi = 0$, where

$$E^2 = \frac{h_1 h_2}{h_3}\left[\partial_{q_1}\left(\frac{h_1 h_3}{h_2}\partial_{q_1}\right) + \partial_{q_2}\left(\frac{h_2 h_3}{h_1}\partial_{q_2}\right)\right].$$

Observe that $E^2 \neq \nabla^2$ in (q_1, q_2, q_3) coordinates, but they are equal in the case of $h_3 = 1$ (or 2D flow?).

Example 6.1.1. Consider the spherical coordinate $(q_1, q_2, q_3) = (r, \theta, \phi)$, where $h_1 = 1, h_2 = \frac{1}{r}, h_3 = \frac{1}{r\sin\theta}$. Then

$$\begin{aligned} u_r &= \frac{1}{r^2\sin\theta}\partial_\theta\psi(r,\theta) \\ u_\theta &= -\frac{1}{r\sin\theta}\partial_r\psi(r,\theta) \\ u_\phi &= 0 \end{aligned}$$

and

$$E^2 = \partial_{rr} + \sin\theta\partial_\theta\left(\frac{1}{r^2\sin\theta}\partial_\theta\right).$$

The spherical Laplacian takes the form

$$\nabla^2 = \frac{1}{r^2}\partial_r\left(r^2\partial_r\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right)$$

Example 6.1.2. Consider the coordinate $(q_1, q_2, q_3) = (r, \eta, \phi)$, where $\eta = \cos\theta \in [-1, 1]$. Then

$$\begin{aligned} u_r &= -\frac{1}{r^2}\partial_\eta \psi \\ u_\eta &= -\frac{1}{r\sqrt{1-\eta^2}}\partial_r \psi \\ u_\phi &= 0 \end{aligned}$$

and

$$E^2 = \partial_{rr} + \frac{1-\eta^2}{r^2}\partial_{\eta\eta}.$$

Example 6.1.3. Consider the cylindrical coordinates $(q_1, q_2, q_3) = (r, z, \phi)$. Then

$$\begin{aligned} u_r &= -\frac{1}{r}\partial_z \psi \\ u_z &= \frac{1}{r}\partial_r \psi. \end{aligned}$$

6.1.4 Flow past a sphere

The Stokes stream function is

$$u_r = \frac{1}{r^2\sin\theta}\partial_\theta \psi, \quad u_\theta = -\frac{1}{r\sin\theta}\partial_r \psi.$$

The no-slip and no-penetration boundary conditions at the surface $r = a$ are

$$u_r = u_\theta = 0 \implies \partial_\theta \psi = \partial_r \psi = 0.$$

The boundary condition at infinity is that $\boldsymbol{u} \longrightarrow U\boldsymbol{e}_z$ as $r \longrightarrow \infty$, which implies that $u_r \sim U\cos\theta$ and $u_\theta \sim -U\sin\theta$ as $r \longrightarrow \infty$. This suggests that $\partial_\theta \psi \sim U\cos\theta\sin\theta r^2$ and $\partial_r \psi \sim U\sin^2\theta r$, or

$$\psi \sim \frac{1}{2}Ur^2\sin^2\theta \quad \text{as } r \longrightarrow \infty.$$

With this in mind, we guess an ansatz of the form

$$\psi(r,\theta) = f(r)\sin^2\theta$$

with $f(r) \sim \frac{1}{2}Ur^2$ as $r \longrightarrow \infty$. Substitute into $E^2\psi$ yields

$$\begin{aligned} E^2\psi &= f''(r)\sin^2\theta + \frac{\sin\theta}{r^2}\partial_\theta\left(\frac{1}{\sin\theta}2f(r)\sin\theta\cos\theta\right) \\ &= f''(r)\sin^2\theta + \frac{\sin\theta}{r^2}2f(r)\left(-\sin\theta\right) \\ &= \left[f''(r) - \frac{2f(r)}{r^2}\right]\sin^2\theta. \end{aligned}$$

Applying E^2 onto $E^2\psi$ we find that

$$E^4\psi = \left[f''(r) - \frac{2f(r)}{r^2}\right]''\sin^2\theta + \frac{\sin\theta}{r^2}\partial_\theta\left(\frac{1}{\sin\theta}\left[f'' - \frac{2f}{r^2}\right]2\sin\theta\cos\theta\right)$$

$$= \left\{ \left[f''(r) - \frac{2f(r)}{r^2} \right]'' - \frac{2}{r^2} \left[f'' - \frac{2f}{r^2} \right] \right\} \sin^2\theta = 0.$$

Since this must be true for any θ, we obtain an ODE for $f(r)$

$$f''''(r) - \frac{8}{r^4} f(r) + \frac{8}{r^3} f'(r) - \frac{4}{r^2} f''(r) = 0,$$

or in a more compact form

$$\left(\frac{d}{dr^2} - \frac{2}{r^2} \right)^2 f(r) = 0.$$

This is a Cauchy-Euler equation so we guess an ansatz of the form $f(r) = r^\alpha$. The general solution is

$$f(r) = \frac{A}{r} + Br + Cr^2 + Dr^4.$$

By virtue of the asymptotic behaviour of $f(r)$ for large r, we must set $D = 0$ and $C = U/2$ and so

$$f(r) = \frac{A}{r} + Br + \frac{1}{2} Ur^2.$$

At $r = a$,

$$\partial_r \psi = f'(r) \sin^2\theta \bigg|_{r=a} = 0 \implies f'(a) = 0$$
$$\partial_\theta \psi = f(r) 2 \sin\theta \cos\theta \bigg|_{r=a} = 0 \implies f(a) = 0.$$

This yields two equations for A, B:

$$0 = \frac{A}{a} + Ba + \frac{1}{2} Ua^2$$
$$0 = -\frac{A}{a^2} + B + Ua$$

which has solution $A = \frac{Ua^3}{4}$ and $B = -\frac{3aU}{4}$. Finally, the Stokes stream function is

$$\psi(r,\theta) = \frac{U}{4} \left(\frac{a^3}{r} - 3ar + 2r^2 \right) \sin^2\theta$$

and the velocity field is

$$\boldsymbol{u}(r,\theta) = \left(\frac{U}{2} \cos\theta \left(\frac{a^3}{r^3} - \frac{3a}{r} + 2 \right), -\frac{U}{4} \sin\theta \left(-\frac{a^3}{r^3} - \frac{3a}{r} + 4 \right), 0 \right).$$

If we can find $p(r,\theta)$, then the stress tensor can be found easily. We are also interested in finding the traction force in the z-direction on the sphere and it can be shown that $F_z = 6\pi\mu aU$. This is called the Stokes drag law. Let $t_z = \boldsymbol{t} \cdot \boldsymbol{e}_z$ be the z-component of the traction force at any point on the surface of the sphere. The traction vector is $\boldsymbol{t} = \underline{\underline{\sigma}} \cdot \boldsymbol{n} = \underline{\underline{\sigma}} \cdot \boldsymbol{e}_r$, where $\underline{\underline{\sigma}}$ is the stress tensor on the surface. In terms of spherical coordinates,

$$\underline{\underline{\sigma}} = \sigma_{rr} \boldsymbol{e}_r \boldsymbol{e}_r + \sigma_{r\theta} \boldsymbol{e}_r \boldsymbol{e}_\theta + \sigma_{r\phi} \boldsymbol{e}_r \boldsymbol{e}_\phi$$

$$+ \sigma_{\theta r}\boldsymbol{e}_\theta\boldsymbol{e}_r + \sigma_{\theta\theta}\boldsymbol{e}_\theta\boldsymbol{e}_\theta + \sigma_{\theta\phi}\boldsymbol{e}_\theta\boldsymbol{e}_\phi$$
$$+ \sigma_{\phi r}\boldsymbol{e}_\phi\boldsymbol{e}_r + \sigma_{\phi\theta}\boldsymbol{e}_\phi\boldsymbol{e}_\theta + \sigma_{\phi\phi}\boldsymbol{e}_\phi\boldsymbol{e}_\phi.$$

Consequently,

$$\underline{\underline{\sigma}} \cdot \boldsymbol{e}_r = \sigma_{rr}\boldsymbol{e}_r + \sigma_{\theta r}\boldsymbol{e}_\theta + \sigma_{\phi r}\boldsymbol{e}_\phi.$$

We also know that $\sigma_{\theta r} = \sigma_{r\theta}$ and $\sigma_{\phi r} = \sigma_{r\phi}$. Recall that the Newtonian stress is

$$\underline{\underline{\sigma}} = -p\underline{\underline{I}} + 2\mu\underline{\underline{E}}.$$

From the expression for $\psi(r,\theta)$, we see that

$$\begin{aligned} u_r &= \frac{1}{r^2\sin\theta}\partial_\theta\psi \; = \frac{U}{2}\cos\theta\left(\frac{a^3}{r^3} - \frac{3a}{r} + 2\right) \\ u_\theta &= -\frac{1}{r\sin\theta}\partial_r\psi = -\frac{U}{4}\sin\theta\left(-\frac{a^3}{r^3} - \frac{3a}{r} + 4\right). \end{aligned}$$

We have yet to determine the pressure p. From the Stokes equation in spherical coordinates (whatttt?),

$$\begin{aligned} \partial_r p &= \frac{\mu}{r^2\sin\theta}\partial_\theta\left(E^2\psi\right) \\ \frac{1}{r}\partial_\theta p &= -\frac{\mu}{r\sin\theta}\partial_r\left(E^2\psi\right). \end{aligned}$$

One can show that $E^2\psi = \dfrac{3Ua\sin^2\theta}{2r}$ [check] and it follows from integrating the system of PDE for p that [check]

$$p(r,\theta) = p_\infty - \frac{3}{2}\frac{\mu U a}{r^2}\cos\theta.$$

We know that $\sigma_{r\phi} = 0$. Referring to the textbook [Acheson/Leal/Batchelor] for the expression of $\underline{\underline{\sigma}}$ in spherical coordinates,

$$\begin{aligned} \sigma_{rr}\Big|_{r=a} &= \left(-p + 2\mu\partial_r u_r\right)\Big|_{r=a} && = -p_\infty + \frac{3}{2}\mu\frac{Ua}{r^2}\cos\theta \\ \sigma_{r\theta}\Big|_{r=a} &= \left(\mu r\partial_r\left(\frac{u_\theta}{r}\right) + \frac{\mu}{r}\partial_\theta u_r\right)\Big|_{r=a} && = -\frac{3\mu U}{a}\sin\theta. \end{aligned}$$

Because $\boldsymbol{e}_r = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and $\boldsymbol{e}_\theta = (\cos\theta\cos\phi, \sin\phi\cos\theta, -\sin\theta)$, we can finally find t_z and integrating this yields the force in the z-direction:

$$\begin{aligned} F_z &= \int_0^{2\pi}\int_0^\pi t_z a^2\sin\theta\, d\theta d\phi \\ &= \int_0^{2\pi}\int_0^\pi \left(\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta\right)a^2\sin\theta\, d\theta d\phi \\ &= 6\pi\mu a U = \gamma U, \end{aligned}$$

where γ is the drag coefficient.

Note that $E^4\psi = 0$ can be viewed as the zeroth-order equation for the asymptotic problem (Navier-Stokes equation), with Reynolds number Re being the small parameter. To get solutions where the convection term $(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}$ can't be neglected,

1. Oseen/Faxen Law (fundamental solutions);see Pozrikidis's book (singular xxxxxxxx)
2. Matched asymptotics (Proudman and Pearson 1957); see Acheson p. 227-228 (Chapter 7).
$$\psi(r,\theta) = \frac{r^2}{4}\sin^2\theta\left[2 + \frac{3}{4}\mathrm{Re}\,(1-\cos\theta) - \frac{3}{r}\right]$$
and the drag coefficient is
$$\gamma = 6\pi\mu a\left(1 + \frac{3}{8}\mathrm{Re}\right).$$

6.2 Solutions via Green's Function

Instead of solving the Stokes equation, we solve the following problem

$$-\nabla p + \mu\Delta\boldsymbol{u} = -\boldsymbol{g}\delta\left(\boldsymbol{x}-\boldsymbol{x}_0\right) \tag{6.2.1}$$
$$\nabla\cdot\boldsymbol{u} = 0 \tag{6.2.2}$$

where the RHS corresponds to a pulse/source at $\boldsymbol{x} = \boldsymbol{x}_0$. Let $\hat{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}_0$. Recall that

$$\delta\left(\hat{\boldsymbol{x}}\right) = -\frac{1}{4\pi}\nabla^2\left(\frac{1}{r}\right), \quad r = \left|\hat{\boldsymbol{x}}\right|.$$

This can be proved using Fourier transform or we integrate over a ball with center $\boldsymbol{x} = \boldsymbol{x}_0$.

Lemma 6.2.1. *The pressure satisfies* $p = -\frac{1}{4\pi}\boldsymbol{g}\cdot\nabla\left(\frac{1}{r}\right)$

Proof. Taking the divergence of (6.2.1) and using the identity

$$\nabla\boldsymbol{u} = \nabla\left(\nabla\cdot\boldsymbol{u}\right) - \nabla\times\left(\nabla\times\boldsymbol{u}\right)$$

yields

$$\begin{aligned}
-\Delta p - \mu\nabla\cdot\left(\nabla\times\nabla\times\boldsymbol{u}\right) &= -\nabla\cdot\left(\boldsymbol{g}\delta\left(\hat{\boldsymbol{x}}\right)\right)\\
-\Delta p &= \boldsymbol{g}\frac{1}{4\pi}\nabla\Delta\left(\frac{1}{r}\right)\\
-p &= \frac{1}{4\pi}\boldsymbol{g}\cdot\nabla\left(\frac{1}{r}\right).
\end{aligned}$$

where we used the identity for the delta function.

■

Substituting the expression for p, we get

$$\begin{aligned}
\frac{1}{4\pi}\nabla\left(\boldsymbol{g}\cdot\nabla\left(\frac{1}{r}\right)\right) + \mu\Delta\boldsymbol{u} &= \frac{\boldsymbol{g}}{4\pi}\Delta\left(\frac{1}{r}\right)\\
\mu\Delta\boldsymbol{u} &= \frac{1}{4\pi}\boldsymbol{g}\Delta\left(\frac{1}{r}\right) - \frac{1}{4\pi}\boldsymbol{g}\cdot\left(\nabla\nabla\left(\frac{1}{r}\right)\right)\\
&= -\frac{1}{4\pi}\boldsymbol{g}\cdot\left(\nabla\nabla - \underline{\underline{I}}\nabla^2\right)\left(\frac{1}{r}\right)
\end{aligned}$$

Take the ansatz

$$\boldsymbol{u} = \frac{1}{\mu}\boldsymbol{g} \cdot \left(\nabla\nabla - \underline{\underline{I}}\nabla^2\right) H(r),$$

where $H(r)$ is a scalar function. Then

$$\nabla^2 H = -\frac{1}{4\pi r}.$$

Applying ∇^2 on each side yields

$$\nabla^4 H = -\frac{1}{4\pi}\nabla^2\left(\frac{1}{r}\right) = \delta\left(\hat{\boldsymbol{x}}\right),$$

i.e. H is the fundamental solution of the biharmonic equation, given by $H = -r/8\pi$. Consequently,

$$\begin{aligned}\boldsymbol{u} &= \frac{1}{\mu}\boldsymbol{g} \cdot \left(\nabla\nabla - \underline{\underline{I}}\nabla^2\right)\left(-\frac{r}{8\pi}\right) \\ &= \frac{1}{8\pi\mu}\left(\frac{1}{r}\underline{\underline{I}} + \frac{1}{r^3}\boldsymbol{x}\boldsymbol{x}^T\right)\boldsymbol{g} \\ &= \frac{1}{8\pi\mu}\underline{\underline{S}}\boldsymbol{g}.\end{aligned}$$

$\underline{\underline{S}}$ is known as the Stokeslet.

Proof.

$$u_i\boldsymbol{e}_i = -\frac{1}{8\pi\mu}g_i\boldsymbol{e}_i \cdot \left(\partial_j\boldsymbol{e}_j\partial_k\boldsymbol{e}_k - \delta_{jk}\boldsymbol{e}_j\boldsymbol{e}_k\partial_{\ell\ell}\right) r.$$

and

$$\partial_i r = \frac{\hat{x}_i}{r}, \quad \partial_i|\boldsymbol{x} - \boldsymbol{x}_0| = \frac{2(x_i - x_{0,i})}{2|\boldsymbol{x} - \boldsymbol{x}_0|}.$$

■

Similarly,

$$\begin{aligned}p &= \frac{1}{4\pi}\left(\frac{\boldsymbol{g}\cdot\hat{\boldsymbol{x}}}{r^3}\right) \\ \underline{\underline{\sigma}} &= -\frac{3}{4\pi}\left(\frac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T}{r^5}\right)\boldsymbol{g}\cdot\hat{\boldsymbol{x}}.\end{aligned}$$

6.3 Boundary integral and singularity methods for linearised viscous flow

See the book with title above by C. Pozrikidis. Let us consider the forced incompressible Stokes equation

$$-\nabla p + \mu\nabla^2\boldsymbol{u} = -\boldsymbol{g}\delta(\boldsymbol{x} - \boldsymbol{x}_0),$$

where $\boldsymbol{g}$ is the strength of the point force located at $\boldsymbol{x}_0$. For notational convenience, we set $\hat{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}_0$. Let $G_{ij}(\boldsymbol{x}, \boldsymbol{x}_0)$ be the Green's function. We can expression $u_i(\boldsymbol{x})$ as

$$u_i(\boldsymbol{x}) = \frac{1}{8\pi\mu} G_{ij}(\boldsymbol{x}, \boldsymbol{x}_0) g_j.$$

The pressure is (do not confuse p_j with p)

$$p(\boldsymbol{x}) = \frac{1}{8\pi} p_j(\boldsymbol{x}, \boldsymbol{x}_0) g_j.$$

The stress tensor is

$$\sigma_{ik}(\boldsymbol{x}) = \frac{1}{8\pi} T_{ijk}(\boldsymbol{x}, \boldsymbol{x}_0) g_j.$$

Example 6.3.1. In the free space,

$$\begin{aligned} G_{ij}(\hat{\boldsymbol{x}}) &= S_{ij}(\hat{\boldsymbol{x}}) = \frac{\partial_{ij}}{r} + \frac{\hat{x}_i \hat{x}_j}{r^3}, \quad r = |\hat{\boldsymbol{x}}| \\ p_j(\hat{\boldsymbol{x}}) &= \frac{2\hat{x}_j}{r^3} \\ T_{ijk}(\hat{\boldsymbol{x}}) &= -\frac{6\hat{x}_i \hat{x}_j \hat{x}_k}{r^5} \end{aligned}$$

Here, $T_{ijk} = -\delta_{ik} p_j + \partial_k G_{ij} + \partial_i G_{kj}$.

Proposition 6.3.2.

$$G_{ij}(\boldsymbol{x}, \boldsymbol{x}_0) = G_{ji}(\boldsymbol{x}, \boldsymbol{x}_0).$$

Proof. Use Lorentz reciprocal theorem. ■

6.3.1 Boundary integral equation

We claim that

$$u_i(\boldsymbol{x}_0) = -\frac{1}{8\pi\mu} \int_D f_i(\boldsymbol{x}) G_{ij}(\hat{\boldsymbol{x}})\, dS(\boldsymbol{x}) + \frac{1}{8\pi} \int_D u_i(\boldsymbol{x}) T_{ijk}(\hat{\boldsymbol{x}}) n_k(\boldsymbol{x})\, dS(\boldsymbol{x})$$

Here, $f_i = \sigma_{ik} n_k$ (surface force???), D is the boundary. Recall the Lorentz reciprocal theorem:

$$\frac{\partial}{\partial x_k} \left(u_i' \sigma_{ik} - u_i \sigma_{ik}' \right) = 0,$$

where $\left(\boldsymbol{u}, \underline{\underline{\sigma}}\right)$ and $\left(\boldsymbol{u}', \underline{\underline{\sigma}}'\right)$ are two solutions of Stokes flow. Let $\left(\boldsymbol{u}', \underline{\underline{\sigma}}'\right)$ be the solution of Stokes flow due to a point force with density $\boldsymbol{g}$ at $\boldsymbol{x}_0$ (free space). Let $\left(\boldsymbol{u}, \underline{\underline{\sigma}}\right)$ be a solution of Stokes flow. We want to find $\boldsymbol{u}(\boldsymbol{x}_0)$. We know everything about $\left(\boldsymbol{u}', \underline{\underline{\sigma}}'\right)$. In particular,

$$\begin{aligned} u_i'(\boldsymbol{x}) &= \frac{1}{8\pi\mu} G_{ij}(\boldsymbol{x}, \boldsymbol{x}_0) g_j \\ \sigma_{ik}'(\boldsymbol{x}) &= \frac{1}{8\pi} T_{ijk}(\boldsymbol{x}, \boldsymbol{x}_0) g_j \end{aligned}$$

Substituting this into the Lorentz reciprocal theorem yields

$$\frac{\partial}{\partial x_k} \left[G_{ij}(\boldsymbol{x}, \boldsymbol{x}_0) \sigma_{ik}(\boldsymbol{x}) - \mu u_i(\boldsymbol{x}) T_{ijk}(\boldsymbol{x}, \boldsymbol{x}_0) \right] = 0$$

Integrating and applying divergence theorem, we have two possible cases:

1. If $\boldsymbol{x}_0 \notin V$, then

$$\int_D \left[G_{ij}\left(\boldsymbol{x}, \boldsymbol{x}_0\right) \sigma_{ik}(\boldsymbol{x}) - \mu u_i(\boldsymbol{x}) T_{ijk}\left(\boldsymbol{x}, \boldsymbol{x}_0\right)\right] n_k(\boldsymbol{x})\, dS(\boldsymbol{x}) = 0.$$

   Here, $\boldsymbol{n}$ is the inward pointing normal of D.

2. Suppose $\boldsymbol{x}_0 \in V$. Let S_ε be the boundary of $B_\varepsilon(\boldsymbol{x}_0)$. On $D_\varepsilon = D \setminus S_\varepsilon$, we have

$$\int_{D_\varepsilon} \left[G_{ij}\left(\boldsymbol{x}, \boldsymbol{x}_0\right) \sigma_{ik}(\boldsymbol{x}) - \mu u_i(\boldsymbol{x}) T_{ijk}\left(\boldsymbol{x}, \boldsymbol{x}_0\right)\right] n_k(\boldsymbol{x})\, dS(\boldsymbol{x}) = 0.$$

   In particular,

$$\int_D \text{integrand}\, dS(\boldsymbol{x}) = -\int_{S_\varepsilon} \text{integrand}\, dS(\boldsymbol{x}).$$

   Let $\varepsilon \longrightarrow 0$, then $\sigma_{ik}(\boldsymbol{x}) \longrightarrow \sigma_{ik}(\boldsymbol{x}_0)$ and $u_i(\boldsymbol{x}) \longrightarrow u_i(\boldsymbol{x}_0)$. To leading order in ε (over S_ε),

$$\begin{aligned} \sigma_{ij} &\approx \frac{\delta_{ij}}{\varepsilon} + \frac{\hat{x}_i \hat{x}_j}{\varepsilon^3} \\ T_{ijk} &\approx -\frac{6\hat{x}_i \hat{x}_j \hat{x}_k}{\varepsilon^5}. \end{aligned}$$

   The RHS becomes

$$\begin{aligned} &-\int_{S_\varepsilon} \left[\left(\frac{\delta_{ij}}{\varepsilon} + \frac{\hat{x}_i \hat{x}_j}{\varepsilon^3}\right) \sigma_{ik}(\boldsymbol{x}) + 6\mu u_i(\boldsymbol{x}) \frac{\hat{x}_i \hat{x}_j \hat{x}_k}{\varepsilon^5}\right] n_k\, dS(\boldsymbol{x}) \\ &= -\int_{S_\varepsilon} \left[\left(\frac{\delta_{ij}}{\varepsilon} + \frac{\hat{x}_i \hat{x}_j}{\varepsilon^3}\right) \sigma_{ik}(\boldsymbol{x}) + 6\mu u_i(\boldsymbol{x}) \frac{\hat{x}_i \hat{x}_j \hat{x}_k}{\varepsilon^5}\right] \left(\frac{\hat{x}_k}{\varepsilon}\right) \varepsilon^2 d\Omega(\boldsymbol{x}) \\ &= -\int_{S_\varepsilon} \left\{\left(\delta_{ij} + \frac{\hat{x}_i \hat{x}_j}{\varepsilon^2}\right) \sigma_{ik}(\boldsymbol{x}) \hat{x}_k + 6\mu u_i(\boldsymbol{x}) \left(\frac{\hat{x}_i \hat{x}_j \hat{x}_k \hat{x}_k}{\varepsilon^4}\right)\right\} d\Omega \end{aligned}$$

   Note that $\hat{x} \longrightarrow 0$ linearly with ε. So as $\varepsilon \longrightarrow 0$, we have

$$\begin{aligned} \delta_{ij} + \frac{\hat{x}_i \hat{x}_j}{\varepsilon^2} &= \delta_{ij} + \mathcal{O}(1) \\ \hat{x}_k &\longrightarrow 0 \\ \sigma_{ik}(\boldsymbol{x}) &\longrightarrow \sigma_{ik}(\boldsymbol{x}_0) \\ u_i(\boldsymbol{x}) &\longrightarrow u_i(\boldsymbol{x}_0) \\ \frac{\hat{x}_i \hat{x}_j \hat{x}_k \hat{x}_k}{\varepsilon^4} &= \mathcal{O}(1) \end{aligned}$$

   First, the second integral becomes

$$-6\mu u_i(\boldsymbol{x}_0) \int_{S_\varepsilon} \frac{\hat{x}_i \hat{x}_j}{\varepsilon^2}\, d\Omega$$

   Also,

$$\int_{S_\varepsilon} \frac{\hat{x}_i \hat{x}_j}{\varepsilon^4}\, dS = \frac{1}{\varepsilon^4} \int_{S_\varepsilon} \hat{x}_i \varepsilon n_j\, dS$$

$$
\begin{aligned}
&= \frac{1}{\varepsilon^3}\int_{V_\varepsilon} \frac{\partial \hat{x}_i}{\partial x_j}\, dS(\boldsymbol{x}) \\
&= \frac{1}{\varepsilon^3}\int_{V_\varepsilon} \frac{\partial \hat{x}_i}{\partial \hat{x}_j}\, dS \\
&= \frac{\delta_{ij}}{\varepsilon^3}\int_{V_\varepsilon} dS(\boldsymbol{x}) \\
&= \frac{\delta_{ij}}{\varepsilon^3}\left(\frac{4}{3}\pi\varepsilon^3\right) \\
&= \frac{4\pi}{3}\delta_{ij}.
\end{aligned}
$$

Finally, the RHS becomes

$$-6\mu u_i(\boldsymbol{x}_0)\frac{4\pi}{3}\delta_{ij}$$

and

$$\int_D \left[G_{ij}\left(\boldsymbol{x}, \boldsymbol{x}_0\right)\sigma_{ik}(\boldsymbol{x}) - \mu u_i(\boldsymbol{x}) T_{ijk}\left(\boldsymbol{x}, \boldsymbol{x}_0\right)\right] n_k(\boldsymbol{x},)\, dS(\boldsymbol{x}) = -8\pi\mu u_j(\boldsymbol{x}_0)$$

which implies that

$$u_j(\boldsymbol{x}_0) = -\frac{8\pi\mu}{\int}{}_D\, G_{ij}\left(\boldsymbol{x}, \boldsymbol{x}_0\right)\sigma_{ik}(\boldsymbol{x}) n_k\, dS(\boldsymbol{x}) + \frac{1}{8\pi}\int_D T_{ijk}\left(\boldsymbol{x}, \boldsymbol{x}_0\right) u_i(\boldsymbol{x}) n_k(\boldsymbol{x})\, dS(\boldsymbol{x}).$$

These are the single-layer potential and double-layer potential.

It can be shown that if

$$\int_D \boldsymbol{u}\cdot\boldsymbol{n}\, dS = 0,$$

then

$$u_j(\boldsymbol{x}_0) = -\frac{1}{8\pi\mu}\int_D q_i(\boldsymbol{x}) G_{ij}\left(\boldsymbol{x}, \boldsymbol{x}_0\right)\, dS(\boldsymbol{x}),$$

where $q_i = f_i - f_i'$.

Example 6.3.3. Let us consider a flow due to an immersed particle.

$$u_j(\boldsymbol{x}_0) = -\frac{1}{8\pi\mu}\int_S G_{ji}\left(\boldsymbol{x}_0, \boldsymbol{x}\right) f_i(\boldsymbol{x})\, dS(\boldsymbol{x}),$$

where $f_i(\boldsymbol{x})$ is the force on S. Consider the Taylor series of G with respect to $\boldsymbol{x}$ around $\boldsymbol{x}_c$, where $\boldsymbol{x}_c$ is in the interior of S. Then This leads to slender body theory and the singularity representation in general.

6.4 Taylor Swimming Sheet

This can be considered as a model of swimming at low Reynolds number. Consider an infinite sheet and a wave travelling down the sheet at speed $c = \omega/k > 0$. Denote the position of any particle of the sheet by

$$(x_s(t), y_s(t)) = (x, a\sin(kx - \omega t)).$$

Note that

$$[a] = \text{length}, \quad [k] = \frac{1}{\text{length}}, \quad [\omega] = \frac{1}{\text{time}}.$$

Let $\lambda = 2\pi/k$ be the wavelength and assume a/λ is small. The particle's velocity is

$$\frac{dy_s}{dt} = -a\omega\cos(kx - \omega t).$$

We claim that the swimming sheet boundary condition gives rise to an oscillatory flow and a steady flow in the x-direction, with speed $U = 2\pi^2 (a/\lambda)^2 c$, provided a/λ is small. We will show this using a regular perturbation series.

Let $\psi(x,y)$ be the stream function (Lagrange), satisfying the biharmonic equation $\nabla^4\psi$. The far-field boundary condition is that the flow is bounded as $y \longrightarrow \infty$. The boundary condition on the sheet is $\boldsymbol{u} = \boldsymbol{u}_s$, *i.e.*

$$\frac{\partial\psi}{\partial y} = 0, \quad \frac{\partial\psi}{\partial x} = a\omega\cos(kx - \omega t) \quad \text{on } y = a\sin(kx - \omega t).$$

A crucial observation is that the time variable only appears on the boundary condition, so we may solve the problem at $t = 0$ and the general solution is obtained by replacing kx by $kx - \omega t$. Defining the dimensionless variables

$$x' = kx, \quad y' = ky, \quad \psi' = \frac{k\psi}{\omega a},$$

where we choose k/a which has dimension of $1/\text{length}^2$ because this will simplify the dimensionless equation. The dimensionless system is

$$\begin{aligned} (\nabla')^4\psi' &= 0 \\ \partial_{y'}\psi' &= 0 \quad \text{on } y' = ka\sin(x') \\ \partial_{x'}\psi' &= \cos(x') \quad \text{on } y' = ka\sin(x') \end{aligned}$$

where ka is the dimensionless parameter which we assume to be small. This follows from assuming a/λ to be small, since

$$\lambda = \frac{2\pi}{k} \implies \varepsilon = \left(\frac{a}{\lambda}\right)\pi \ll 1.$$

We may now drop all the primes to simplify notation.

Let us expand the boundary condition around $y = 0$:

$$\begin{aligned} 0 &= \partial_y\psi|_{y=0} + y\partial_{yy}\psi|_{y=0} + \ldots\ldots \\ &= \partial_y\psi|_{y=0} + \varepsilon\sin(x)\partial_{yy}\psi|_{y=0} + \ldots\ldots \\ \cos(x) &= \partial_x\psi|_{y=0} + y\partial_{yx}\psi|_{y=0} + \ldots\ldots \\ &= \partial_x\psi|_{y=0} + \varepsilon\sin(x)\partial_{yx}\psi|_{y=0} + \ldots\ldots\ldots \end{aligned}$$

We assume a regular perturbation series of the form

$$\psi(x,y) \sim \psi_1(x,y) + \varepsilon\psi_2(x,y) + \ldots\ldots \quad \text{as } \varepsilon \longrightarrow 0.$$

The leading-order equation is

$$\begin{aligned} \nabla^4\psi_1 &= 0 && \text{for } y>0 \\ \partial_y\psi_1 &= 0 && \text{on } y=0 \\ \partial_x\psi_1 &= \cos x && \text{on } y=0 \end{aligned}$$

With the given boundary condition, we guess an ansatz of the form $\phi_1(x,y) = f(y)\sin x$.

$$\begin{aligned} \nabla^4\psi_1 &= (\partial_{xxxx} + \partial_{yyyy} + 2\partial_{xxyy})\,\psi_1 \\ &= f(y)\sin x + f''''(y)\sin x - 2f''(y)\sin x \\ &= [f''''(y) - 2f''(y) + f(y)]\sin x = 0. \end{aligned}$$

The general solution for $f(y)$ is

$$f(y) = (A+By)\,e^{-y} + (C+Dy)\,e^{y}.$$

We must have $C = D = 0$ in order that the velocity be bounded as $y \longrightarrow \infty$ and so

$$\psi_1(x,y) = (A+By)\,e^{-y}\sin x.$$

Since

$$\partial_y\psi_1 = \sin x e^{-y}\left(B - (A+By)\right),$$

imposing the boundary conditions yield $B = A$ and $A = 1$. Hence,

$$\psi_1(x,y) = (1+y)\,e^{-y}\sin x.$$

The $\mathcal{O}(\varepsilon)$ equation is

$$\begin{aligned} \nabla^4\psi_2 &= 0 && \text{for } y>0 \\ \partial_y\psi_2 + \sin x\partial_{yy}\psi_1 &= 0 && \text{on } y=0 \\ \partial_x\psi_2 + \sin x\partial_{yx}\psi_1 &= 0 && \text{on } y=0 \end{aligned}$$

Let us compute the required derivative of ψ_1 and evaluate them at $y = 0$:

$$\begin{aligned} \partial_y\psi_1 &= \sin x\left(-ye^{-y}\right) \\ \partial_{yy}\psi_1 &= \sin x\left(-e^{-y} + ye^{-y}\right) \\ \partial_{yx}\psi_1 &= \cos x\left(-ye^{-y}\right) \end{aligned}$$

Consequently, the $\mathcal{O}(\varepsilon)$ equation is

$$\begin{aligned} \nabla^4\psi_2 &= 0 && \text{for } y>0 \\ \partial_y\psi_2 - \sin^2 x &= 0 && \text{on } y=0 \\ \partial_x\psi_2 &= 0 && \text{on } y=0 \end{aligned}$$

Using $\cos(2x) = 1 - 2\sin^2 x$, we guess an ansatz of the form $\phi_2(x,y) = f_1(y) + f_2(y)\cos(2x)$. The general solution for $f_1(y)$ is

$$f_1(y) = Ay^3 + By^2 + Cy + D.$$

Since we want $f_1'(y)$ to be bounded as $y \longrightarrow \infty$, we must have $A = B = 0$. Because the additive constant D has no significance, we may set $D = 0$ and so $f_1(y) = Cy$. A similar argument as above shows that the general solution of $f_2(y)$ is

$$f_2(y) = (E + Fy)\, e^{-2y} + (G + Hy)\, e^{2y}.$$

We must have $G = H = 0$ again to have bounded solution. Summarising altogether, we find that

$$\psi_2(x, y) = Cy + (E + Fy)\, e^{-2y} \cos(2x).$$

Imposing the boundary condition gives $E = 0$, $C = 1/2$ and $F = -1/2$. Finally,

$$\psi_2(x, y) = \frac{y}{2} - \frac{y}{2} e^{-2y} \cos(2x).$$

We now prove the desired expression for the steady flow speed U. Converting back to the dimensional variables, we find that

$$\psi(x, y) = \frac{\omega a}{k} \left[(1 + ky)\, e^{-ky} \sin(kx) + ka \left(\frac{ky}{2} - \frac{ky}{2} e^{-ky} cos(2kx) \right) + \ldots\ldots \right]$$

and

$$\begin{aligned} u = \partial_y \psi &= \frac{\omega a}{k} \left[-k^2 e^{-ky} \sin(kx) + \varepsilon \left(\frac{k}{2} + \left(-\frac{k}{2} + k^2 y \right) e^{-2ky} \cos(2kx) \right) + \ldots\ldots \right] \\ &= -\omega k a y e^{-ky} \sin(kx) + \varepsilon \omega a \left(\frac{1}{2} + \left(ky - \frac{1}{2} \right) e^{-2ky} \cos(2kx) \right) + \ldots\ldots \\ &= -\varepsilon \omega y e^{-ky} \sin(kx) + \frac{\varepsilon^2 c}{2} + \varepsilon^2 c \left(ky - \frac{1}{2} \right) e^{-2ky} \cos(2kx) + \ldots\ldots. \end{aligned}$$

The steady flow at ∞ is

$$U = \frac{\varepsilon^2 c}{2} = \frac{k^2 a^2 c}{2} = 2\pi^2 \left(\frac{a}{\lambda} a \right)^2 c.$$

6.5 Rigid Bodies

Consider solving the Stokes flow past a rigid body V_p, with surface S_p, subject to the condition that

$$\boldsymbol{u} = \underbrace{\boldsymbol{U}}_{\text{translation}} + \underbrace{\boldsymbol{\Omega} \times \boldsymbol{x}}_{\text{rotation}} \quad \text{on } S_p$$

and

$$\boldsymbol{u} \longrightarrow \boldsymbol{0} \quad \text{as } |\boldsymbol{x}| \longrightarrow \infty$$

Suppose the motion is purely translational, *i.e.* $\boldsymbol{\Omega} = \boldsymbol{0}$. Consider the force on the body

$$\boldsymbol{F} = \int_{S_p} \underline{\underline{\sigma}} \cdot \boldsymbol{n}\, dS$$

and the torque on the body

$$\boldsymbol{L} = \int_{S_p} \boldsymbol{x} \times \left(\underline{\underline{\sigma}} \cdot \boldsymbol{n} \right) dS.$$

Since the solutions to the Stokes flow are linear, $\boldsymbol{F}$ and $\boldsymbol{L}$ are linear in $\boldsymbol{U}$. Consequently, we can write $\boldsymbol{F} = \underline{\underline{A}}\boldsymbol{U}$ and $\boldsymbol{L} = \underline{\underline{C}}\boldsymbol{U}$, where $\underline{\underline{A}}, \underline{\underline{C}}$ are the resistance matrix and $\underline{\underline{A}}^{-1}$ the mobility matrix. The matrix $\underline{\underline{C}}$ is not symmetric by virtue of the cross-product in $\boldsymbol{L}$.

$$\begin{aligned} \text{true vector} &= \text{contravariant vector (vector image)} \\ \text{pseudovector} &= \text{gain an extra minus under reflection} \end{aligned}$$

If $\boldsymbol{U} = \boldsymbol{e}_1$, then $F_j = A_{j1}, j = 1, 2, 3$. So $\underline{\underline{A}}$ can be computed for a particular body by solving 3 translational problems. In general,

$$\begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{L} \end{bmatrix} = \begin{bmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \underline{\underline{C}} & \underline{\underline{D}} \end{bmatrix} \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{\Omega} \end{bmatrix}$$

with $\underline{\underline{A}}, \underline{\underline{D}}$ symmetric and $\underline{\underline{B}} = \underline{\underline{C}}^T$. The expressions for resistance matrices can be found as follows:

1. Solve the PDE problem with $\boldsymbol{\Omega} = \boldsymbol{0}$ and then $\boldsymbol{U} = \boldsymbol{0}$.

2. See "Low Reynolds number hydrodynamics" by Happel and Brenner, 1973.

Example 6.5.1 (See Happel and Brenner)**.** For a rigid body with spherical symmetry,

$$\underline{\underline{A}} = \mu a \underline{\underline{I}}, \quad \underline{\underline{D}} = \mu d \underline{\underline{I}}, \quad \underline{\underline{C}} = \underline{\underline{B}} = \boldsymbol{0}.$$

Here, $a = 6\pi r$. For ellipsoid,

$$\underline{\underline{A}} = \mu \begin{bmatrix} a_{\parallel} & 0 & 0 \\ 0 & a_{\perp} & 0 \\ 0 & 0 & a_{\perp} \end{bmatrix}.$$

One example of rigid body with nonzero $\underline{\underline{B}}, \underline{\underline{C}}$ is corkscrew, flagella, helical body?

6.6 Singularity Method

[See Pozrikidis] Let's us tackle the translating sphere problem using a different approach: singularities approach. The problem is

$$\begin{aligned} -\nabla p + \mu \Delta \boldsymbol{u} &= 0 && \text{in } V_p \\ \nabla \cdot \boldsymbol{u} &= 0 && \text{in } V_p \\ \boldsymbol{u} &= \boldsymbol{U} && \text{on } S_p \\ \boldsymbol{u} &\longrightarrow \boldsymbol{0} && \text{as } |x| \longrightarrow \infty \end{aligned}$$

We consider the list of singularities:

1. Point force: Stokeslet $\underline{\underline{G}} = \underline{\underline{I}}/r + \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T/r^3$, $\hat{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}_0$. Differentiating with respect to $\boldsymbol{x}_0$ gives the Doublet or point force dipole, denoted by

$$G^D_{ijk} = \frac{\partial G_{ij}}{\partial x_{0,k}}.$$

We can split this into its symmetric G_{ijk}^{D-S} and anti-symmetric parts G_{ijk}^{D-A}. Also,

$$G_{ijk}^{D-S} = -\delta_{ik}\sigma_j + G_{ijk}^{STR},$$

where

$$G_{ijk}^{STR} = \frac{3\hat{x}_i\hat{x}_j\hat{x}_k}{r^3} \text{ is the stresslet.}$$

and σ_j the point source (potential, rank one tensor). The antisymmetric part is

$$G_{ijk}^{D-A} = \frac{\delta_{ij}\hat{x}_k - \delta_{ik}\hat{x}_j}{r^3}$$

and we can define the rotlet/couplet G_{ij}^C, given by

$$G_{ij}^C = \varepsilon_{ij\ell}\frac{\hat{x}_\ell}{r^3} = -\frac{1}{2}\varepsilon_{kjm}G_{ikj}^{D-A}.$$

2. Point source- potential flow, with constant pressure. The irrotational solution is $\Phi = -\frac{1}{4\pi r}$ and $\boldsymbol{u} = s\boldsymbol{\Sigma}$, where $\boldsymbol{\Sigma} = \hat{\boldsymbol{x}}/r^3$ (s is the strength). Differentiating with respect to $\boldsymbol{x}_0$ leads to potential dipole and quadrupole. The potential dipole is

$$D_{ij} = -\frac{\partial\Sigma_i}{\partial x_{0,j}} = -\frac{\delta_{ij}}{r^3} + \frac{3\hat{x}_i\hat{x}_j}{r^5},$$

and it induces the flow field $u_i = D_{ij}d_j$.

We want to express $\boldsymbol{u}$ as a sum of singularities located inside the sphere. In general,

$$u_i(\boldsymbol{x}) = G_{ij}(\boldsymbol{x},\boldsymbol{x}_0)g_j + G_{ijk}^D c_{jk} + s\Sigma_i + D_{ij}d_j + \text{some other singularities.}$$

It turns out that we just need

$$u_i(\boldsymbol{x}) = G_{ij}(\boldsymbol{x},\boldsymbol{x}_0)g_j + D_{ij}d_j.$$

To satisfy the boundary conditions,

$$\begin{aligned}\boldsymbol{u} = \boldsymbol{U} &= \left(\left(\frac{1}{a}\boldsymbol{g} + \frac{\hat{x}\hat{x}^T}{a^3}\boldsymbol{g}\right)\right) + \left(-\frac{1}{a^3}\boldsymbol{d} + \frac{3\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T}{a^3}\boldsymbol{d}\right) \\ &= \left(\frac{1}{a}\boldsymbol{g} - \frac{1}{a^3}\boldsymbol{d}\right) + \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T\left(\frac{1}{a^3}\boldsymbol{g} + \frac{1}{a^5}\boldsymbol{d}\right).\end{aligned}$$

Solving gives

$$\boldsymbol{g} = \frac{3a}{4}\boldsymbol{U} \quad \text{and} \quad \boldsymbol{d} = -\frac{a^3}{4}\boldsymbol{U}$$

and

$$\boldsymbol{u} = \left(\frac{1}{r}\underline{\underline{I}} + \frac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T}{r^3}\right)\frac{3a}{4}\boldsymbol{U} - \left(\frac{tenI}{r^3} + 3\frac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T}{r^5}\right)\frac{a^3}{4}\boldsymbol{U}.$$

In particular,

See also [Blake and Chuang 1974].

Chapter 7

Elastic instabilities of polymer solutions

Recall the Oldroyd-B Model:

$$
\begin{aligned}
\rho\left(\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) &= -\nabla p + \eta_s\Delta\boldsymbol{u} + \nabla\cdot\underline{\underline{\tau}}_p + \boldsymbol{f}\\
\nabla\cdot\boldsymbol{u} &= 0\\
\lambda\underline{\underline{\tau}}_p + \underline{\underline{\tau}}_p &= 2\eta_p\underline{\underline{D}}
\end{aligned}
$$

where η_s is the solvent viscosity, η_p the polymer viscosity and λ is the elastic relaxation time. Also, $\underline{\underline{D}}$ is the rate of deformation tensor. The notation $\overset{\triangledown}{\cdot}$ refers to the upper convected Maxwell derivative, given as

$$
\overset{\triangledown}{\underline{\underline{\tau}}}_p = \partial_t\underline{\underline{\tau}}_p + \boldsymbol{u}\cdot\underline{\underline{\tau}}_p - \nabla\boldsymbol{u}\underline{\underline{\tau}}_p - \left(\nabla\boldsymbol{u}\underline{\underline{\tau}}_p\right)^T.
$$

We adopt the convention here that $(\nabla\boldsymbol{u})_{ij} = \partial_j u_i$. One interesting quantity is $G = \eta_p/\lambda$, the polymer elastic modulus. Note that we can rewrite the stress tensor equation as

$$
\overset{\triangledown}{\underline{\underline{\tau}}}_p = 2G\underline{\underline{D}} - \frac{1}{\lambda}\underline{\underline{\tau}}_p.
$$

Let $\underline{\underline{\sigma}}_p = \underline{\underline{\sigma}}_p + G\underline{\underline{\delta}}$. Using the fact that

$$
\overset{\triangledown}{\underline{\underline{\delta}}} = -\nabla\boldsymbol{u}\underline{\underline{\delta}} - \left(\nabla\boldsymbol{u}\underline{\underline{\delta}}\right)^T = -2\underline{\underline{D}},
$$

we can show that

$$
\lambda\overset{\triangledown}{\underline{\underline{\sigma}}}_p + \underline{\underline{\sigma}}_p = G\underline{\underline{\delta}}.
$$

The stress creation here is isotropic and $\underline{\underline{\sigma}}_p$ decays with rate $1/\lambda$. Note also that $\nabla\cdot\underline{\underline{\tau}}_p = \nabla\cdot\underline{\underline{\sigma}}_p$. Define the following dimensionless variables

$$
\boldsymbol{u} = U\boldsymbol{u}', \quad \boldsymbol{x} = L\boldsymbol{x}', \quad t = Tt', \quad p = Pp', \quad \underline{\underline{\sigma}}_p = \Sigma\underline{\underline{\sigma}}'_p, \quad \boldsymbol{f} = F\boldsymbol{f}'.
$$

and $U = L/T$, T is a fluid time-scale. Then

$$
\mathrm{Re}\left(\boldsymbol{u}'_{t'} + \boldsymbol{u}'\cdot\nabla'\boldsymbol{u}'\right) = -\frac{PT}{\eta_s}\nabla'p' + \Delta'\boldsymbol{u}' + \left(\frac{\Sigma T}{\eta_s}\right)\nabla'\cdot\underline{\underline{\sigma}}'_p + \left(\frac{\rho FTL}{\eta_s}\right)\boldsymbol{f}'
$$

$$\nabla' \cdot \boldsymbol{u}' = 0$$

$$\partial_{t'} \underline{\underline{\sigma}}'_p + \boldsymbol{u}' \cdot \nabla' \underline{\underline{\sigma}}'_p - \nabla' \boldsymbol{u}' \underline{\underline{\sigma}}'_p - \left(\nabla' \boldsymbol{u}' \underline{\underline{\sigma}}'_p\right)^T = \left(\frac{GT}{\lambda\Sigma}\right) \underline{\underline{\delta}} - \frac{T}{\lambda} \underline{\underline{\sigma}}'_p$$

where $\mathrm{Re} = \rho U L / \eta_s$. We choose T such that

$$1 = \frac{\rho F T L}{\eta_s} \iff T = \frac{\eta_s}{\rho F L}.$$

We choose P such that

$$1 = \frac{PT}{\eta_s} \iff P = \frac{\eta_s}{T} = \rho F L.$$

We choose $\Sigma = G$. Define the Weissenberg number $\mathrm{Wi} = \lambda / T$ as ratio of the elastic relaxation time over the time scale of the fluid motion. Define $\beta = GT/\eta_s$ (contribute to the stress). Consequently,

$$\mathrm{Re}\left(\boldsymbol{u}'_{t'} + \boldsymbol{u}' \cdot \nabla' \boldsymbol{u}'\right) = -\nabla' p' + \Delta' \boldsymbol{u}' + \beta \nabla' \cdot \underline{\underline{\sigma}}'_p + \boldsymbol{f}'$$

$$\nabla' \cdot \boldsymbol{u}' = 0$$

$$\partial_{t'} \underline{\underline{\sigma}}'_p + \boldsymbol{u}' \cdot \nabla' \underline{\underline{\sigma}}'_p - \nabla' \boldsymbol{u}' \underline{\underline{\sigma}}'_p - \left(\nabla' \boldsymbol{u}' \underline{\underline{\sigma}}'_p\right)^T = \frac{1}{\mathrm{Wi}} \left(\underline{\underline{\delta}} - \underline{\underline{\sigma}}'_p\right)$$

Let $\underline{\underline{S}} = \underline{\underline{\sigma}}_p$ and $\underline{\underline{I}} = \underline{\underline{\delta}}$. Setting $\mathrm{Re} = 0$, we obtain the (dimensionless) Stokes Oldroyd-B model

$$0 = -\nabla p + \Delta \boldsymbol{u} + \beta \nabla \cdot \underline{\underline{S}} + \boldsymbol{f}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\overset{\nabla}{\underline{\underline{S}}} = -\frac{1}{\mathrm{Wi}} \left(\underline{\underline{S}} - \underline{\underline{I}}\right)$$

The dimensionles number β is the relative influence of polymer stress on fluid motion. For large Wi, we see that $\underline{\underline{S}}$ "adjusts slowly". Large β means larger elastic force. Observe that

$$\mathrm{Wi}\beta = \frac{GT}{\eta_s} \frac{\lambda}{T} = \frac{G\lambda}{\eta_s} = \frac{\eta_p}{\eta_s}.$$

i.e. $\mathrm{Wi}\beta$ equals the ratio of polymer to solvent viscosity. In the experiment by Arratia, they choose $\eta_p/\eta_s = 1/2$ and in this case we are left with one dimensionless quantity Wi (This is used in many of Becky's simulations). The computational domain is the doubly periodic domain $[0, 2\pi]^2$ with periodic boundary conditions. It can be shown that the equation for $\underline{\underline{S}}$ preserves symmetric positive-definiteness of $\underline{\underline{S}}$, but what about the numerics?

To generate stress both upward and downwards, we can modify the flow field $\boldsymbol{u} = \alpha(y, x)$ (the stagnation flow) that has hyperbolic flow. We choose $\boldsymbol{f}$ so that without polymer the velocity is

$$\boldsymbol{u}(x, y) = \begin{bmatrix} -\sin x \cos y \\ \cos x \sin y \end{bmatrix}.$$

This creates hyperbolic points at background flow. One can check that the background force is

$$\boldsymbol{f}(x, y) = \begin{bmatrix} 2 \sin x \cos y \\ -2 \cos x \sin y \end{bmatrix}.$$

Choose the initial condition $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{0}$ and $\underline{\underline{S}}(\boldsymbol{x}, 0) = \underline{\underline{I}}$. The vorticity (in $2D$) is a scalar so it's easier to plot.

7.1 Evolution of S_{11}

The equation is

$$(S_{11})_T + \varepsilon x (S_{11})_x - \varepsilon y (S_{11})_y + (1-2\varepsilon) S_{11} - 1 = 0 \tag{7.1.1}$$

We can solve using the method of characteristics:

$$\begin{aligned} \frac{dT}{ds} &= 1 \implies T = s \\ \frac{dx}{ds} &= \varepsilon x \implies x(s) = x_0 e^{\varepsilon s} \\ \frac{dy}{ds} &= -\varepsilon y \implies y(s) = y_0 e^{-\varepsilon s} \end{aligned}$$

Let $\tilde{S}_{11}(s) = S_{11}(x(s), y(s), t)$. Then

$$\frac{d\tilde{S}}{ds} + (1-2\varepsilon)\tilde{S}_{11} - 1 = 0.$$

The solution is

$$\tilde{S}_{11}(s) = \frac{1}{1-2\varepsilon} + \left(\tilde{S}_{11}(s=0) - \frac{1}{1-2\varepsilon}\right) e^{(2\varepsilon-1)s},$$

where

$$\tilde{S}_{11}(s=0) = S_{11}(x_0, y_0, 0).$$

The general solution can be written as

$$\begin{aligned} S_{11}(x,y,T) &= \frac{1}{1-2\varepsilon} + H_{11}(x_0, y_0) e^{(2\varepsilon-1)s} \\ &= \frac{1}{1-2\varepsilon} + H_{11}\left(x e^{-\varepsilon T}, y e^{\varepsilon T}\right) e^{(2\varepsilon-1)T} \end{aligned}$$

with H_{11} arbitrary.

Firstly, take $H_{11} = h_{11}\left(y e^{\varepsilon T}\right)$ (gradual dependence on x) and suppose

$$h_{11}\left(y e^{\varepsilon T}\right) \sim \left|y e^{\varepsilon T}\right|^q \quad \text{for } \varepsilon T \gg 1.$$

In fact, for concreteness we take

$$h_{11}\left(y e^{\varepsilon T}\right) \sim h_0 \Big(1 + C y^2 e^{2\varepsilon T}\Big)^{q/2}.$$

For $\varepsilon T \gg 1$,

$$S_{11} \sim \frac{1}{1-2\varepsilon} + h_0 \Big(1 + C y^2 e^{2\varepsilon T}\Big)^{q/2} e^{(2\varepsilon-1)T}$$

To obtain a time-independent expression for large $\varepsilon T \gg 1$, we choose q such that

$$q\varepsilon + (2\varepsilon - 1) = 0 \implies q = \frac{1-2\varepsilon}{\varepsilon}.$$

This leads to

$$S_{11} \sim S_{11}^{\infty} = \frac{1}{1-2\varepsilon} + A|y|^{(1-2\varepsilon)/\varepsilon} \quad \text{for } \varepsilon T \gg 1.$$

How does S_{11}^{∞} behave for different values of q? For $q < 0$, *i.e.* $\varepsilon > 1/2$, we see that $|y|^q \longrightarrow \infty$ as $|y| \longrightarrow 0$ (singularity at $y = 0$). For $0 < q < 1$, *i.e.* $1/3 < \varepsilon < 1/2$, we get a cusp for $|y|^q$, derivative blows up as $y \longrightarrow 0$. Note that the singularity is integrable for $-1 < q < 0$.

Recall that

$$\begin{aligned} -\nabla p + \Delta \boldsymbol{u} &= -\beta \nabla \cdot \underline{\underline{S}} + \boldsymbol{f} \\ \overset{\nabla}{\underline{\underline{S}}} &= -\frac{1}{\mathrm{Wi}} \left(\underline{\underline{S}} - \underline{\underline{I}} \right). \end{aligned}$$

7.2 Spectral Methods

These are a subset of weighted residual methods, which also includes the finite element methods. Suppose we are interested in $x \in [\alpha, \beta]$ and we have a weighted inner product

$$(u, v)_w = \int_\alpha^\beta u(x)v(x)w(x)\, dx,$$

where $w(x) \geq 0$ is called a weight function. Suppose we have a set of trial functions $\{\phi_k(x), k = 0, 1, \dots\}$ which are orthogonal with respect to $(\cdot, \cdot)_w$. The most important examples in spectral methods are

1. trigonometry functions e^{ikx} for periodic problems;

2. Chebyshev polynomials $T_k(x)$ for nonperiodic problems.

In both cases, $(\phi_k, \phi_j)_w = \delta_{jk}$ with appropriate weight functions (constant for trig functions, $1/\sqrt{1 - x^2}$ for Chebyshev, check this). Consider

$$u_N(x) = \sum_{k=0}^{N} \hat{u}_k \phi_k(x).$$

We consider two problems here:

1. $u_N(x)$ to approximate a given $u(x)$, or

2. $u_N(x)$ to approximate the solution $u(x)$ of a differential equation $Lu = f$.

The residual $R_N(x)$ is

$$R_N(x) = u(x) - u_N(x)$$

for the first problem and

$$R_N(x) = Lu_N(x) - f(x)$$

for the second problem. The method of weighted residual makes $R_N(x)$ zero approximately by setting to 0 the inner products

$$(R_N, \Psi_j)_{w_*} = \int_\alpha^\beta R_N(x)\Psi_j(x)w_*(x)\, dx$$

where $\{\Psi_j, j = 0, 1, \dots\}$ are called test functions and the weight function $w_*(x)$ need not be the same as $w(x)$. There are two broad classes of methods.

1. Galerkin methods: $\Psi_j(x) = \phi_j(x)$ and $w_* = w$. In this case,

$$0 = (R_N, \phi_j)_w = \int_\alpha^\beta R_N(x)\phi_j(x)w(x)\,dx = 0,$$

   i.e. we require the residual to be zero in the weighted L^2 sense.

2. Collocation methods: $\Psi_j(x) = \delta(x - x_j)$, with $x_0, x_1, \ldots, x_N$ the collocation points, and $w_*(x) \equiv 1$. In this case,

$$(R_N, \Psi_j)_{w_*} = \int_\alpha^\beta R_N(x)\delta(x - x_j)\,dx = R_N(x_j),$$

   i.e. we require the residual to be zero on a finite set of collocation points.

Let us consider the approximation problem. Using the Galerkin approach,

$$u_N(x) = \sum_{k=0}^{N} \hat{u}_k \phi_k(x)$$
$$R_N(x) = u(x) - \sum_{k=0}^{N} \hat{u}_k \phi_k(x)$$

We want to choose the coefficients $\{\hat{u}_k\}$ such that

$$0 = (R_N(x), \phi_j(x)) = \int_\alpha^\beta \left(u(x) - \sum_{k=0}^{N} \hat{u}_k \phi_k(x) \right) \phi_j(x) w(x)\,dx$$
$$\sum_{k=0}^{N} (\phi_k, \phi_j)_w \hat{u}_k = (u, \phi_j)_w \quad \text{for all } j = 0, 1, \ldots, N.$$

Since $(\phi_k, \phi_j)_w = \delta_{kj} c_k$, we have that

$$c_j \hat{u}_j = (u, \phi_j)_w \implies \hat{u}_j = \frac{(u, \phi_j)_w}{(\phi_j, \phi_j)_w} \quad \text{for all } j = 0, 1, \ldots, N.$$

7.3 LAST DAY

Consider the viscous Burger's equation on a 2π-periodic domain

$$u_t + u u_x = \beta u_{xx}$$
$$u(x, 0) = u_0(x)$$

Consider the set of trigonometric trial functions $\phi_k(x) = e^{ikx}, k = 0, 1, \ldots \ldots$. With the inner product

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\overline{g(x)}\,dx,$$

we have the following orthogonality relation:

$$(\phi_k, \phi_l) = 2\pi \delta_{kl}.$$

Define the Fourier coefficient

$$\hat{u}_k = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-ikx} \, dx \tag{7.3.1}$$

and the Fourier series

$$P_N u(x,t) = \sum_{k=-N/2}^{N/2-1} \hat{u}_k e^{ikx} \tag{7.3.2}$$

$$Su(x,t) = \sum_{k=-\infty}^{\infty} \hat{u}_k(t) e^{ikx}$$

Note that $P_N u$ is the orthogonal projection of u onto $S_N = \operatorname{span}\{\phi_{-\frac{N}{2}}, \phi_{-\frac{N}{2}+1}, \ldots, \phi_{\frac{N}{2}-1}\}$. Let us define the index set

$$\begin{aligned} K_N &= \left\{ k\colon\; -\frac{N}{2} \le k \le \frac{N}{2} - 1 \right\} \\ K_N' &= \left\{ k\colon k < -\frac{N}{2} \;\text{ or }\; k > \frac{N}{2} - 1 \right\}. \end{aligned}$$

If $u \in L^2(0, 2\pi)$, then $P_N u \longrightarrow u$ in L^2 and

$$\begin{aligned} \|u\|_2^2 &= 2\pi \sum_k |\hat{u}_k|^2 \\ \|u - P_n u\|_2^2 &= 2\pi \sum_{K_N'} |\hat{u}_k|^2 . \end{aligned}$$

For sufficiently smooth u,

$$\|u - P_N u\|_\infty \le \sum_{K_N'} |\hat{u}_k|.$$

If u is m-times continuously differentialbe with $u', u'', \ldots, u^{(m-2)}$ 2π-periodic, then

$$|\hat{u}_k| = \mathcal{O}\left(\frac{1}{|k|^m} \right).$$

If $u \in C^\infty$ and u plus all its derivatives are 2π-periodic, then the decay estimate holds for all $m \in \mathbb{Z}^+$, *i.e.* the Fourier coefficient decays to 0 faster than $|k|^{-m}$ for every $m \in \mathbb{Z}^+$. This is known as spectral convergence.

Define the Fourier nodes $x_j = 2\pi j/N, j = 0, 1, \ldots, N$. Define the discrete Fourier transform

$$\tilde{u}_k = \frac{1}{N} \sum_{j=0}^{N-1} u(x_j) e^{-ikx_j}, \quad k \in K_N. \tag{7.3.3}$$

We have the orthogonality relation:

$$\frac{1}{N} \sum_{j=0}^{N-1} e^{ipx_j} = \begin{cases} 1 & \text{if } p = mN, m \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

The inverse discrete Fourier transform is

$$u(x_j) = \sum_{k\in K_N} \tilde{u}_k e^{ikx_j}$$

and this is exact for all the Fourier nodes, in the sense that if

$$I_N u(x) = \sum_{k\in K_N} \tilde{u}_k e^{ikx}, \tag{7.3.4}$$

then $I_N u(x_j) = u(x_j)$ for all $j = 0, 1, \ldots, N-1$. I_N can be thought of as the interpolant of u at nodes. We point out that (7.3.3) can be viewed as approximating the integral (7.3.1) using trapezoidal rule with nodes $x_1, \ldots, x_N$. $\tilde{u}_k$ and $\{\hat{u}_\ell\}$ can be related as follows:

$$\begin{aligned}
\tilde{u}_k = \frac{1}{N}\sum_{j=0}^{N-1} u(x_j)e^{-ikx_j} &= \frac{1}{N}\sum_{j=0}^{N-1}\left(\sum_{\ell}\hat{u}_\ell e^{i\ell x_j}\right)e^{-ikx_j} \\
&= \sum_{\ell}\hat{u}_\ell\left(\frac{1}{N}\sum_{j=0}^{N-1} e^{i(\ell-k)x_j}\right) \\
&= \hat{u}_k + \hat{u}_{k+N} + \hat{u}_{k-N} + \hat{u}_{k+2N} + \hat{u}_{k-2N} + \ldots\ldots \\
&= \hat{u}_k + \sum_{m=\mathbb{Z}\setminus\{0\}} \hat{u}_{k+mN}, \quad k \in K_N.
\end{aligned}$$

Note that $e^{i(k+mN)x}$ aliases e^{ikx}, *i.e.* they are indistinguishable about the Fourier nodes. In terms of the inverse discrete Fourier transform,

$$I_N u(x) = P_N u(x) + A_N u(x),$$

where

$$A_N u(x) = \sum_{k\in K_N}\left(\sum_{m\in\mathbb{Z}\setminus\{0\}} \hat{u}_{k+mN}\right)\phi_k(x)$$

is the aliasing error in representing u in terms of a linear combination of $\{\phi_k, k \in K_N\}$. One can show that $(u - P_N u) \perp A_N u$

$$\|u - I_N u\|_2^2 = \|u - P_N u\|_2^2 + \|A_N u\|_2^2,$$

i.e. the error using the interpolant is greater than the error using the truncated series. It can also be shown that asymptotically as $|k| \longrightarrow \infty$ that $\|u - P_N u\|_2$ and $\|A_N u\|_2$ decay at the same rate.

7.3.1 Differentiation with respect to x

For the Fourier series $Su(x)$, its derivative is

$$S'u(x) = \sum_{k=-\infty}^{\infty} ik\hat{u}_k e^{ikx}. \tag{7.3.5}$$

This follows from the fact that differentiation and the truncated Fourier series commute, *i.e.* $\Big[P_N u(x)\Big]' = \Big[P_N(u'(x))\Big]$. $(P_N u)'$ is called the Fourier Galerkin derivative.

On the physical space, differentiation depends on the nodal value $u(x_j), j = 0, 1, \ldots, N-1$. We define the Fourier collocation derivative

$$(\mathcal{D}_N u)(x_\ell) = \sum_{k \in K_N} a_k e^{ikx_\ell} = (I_N u)'(x_\ell), \tag{7.3.6}$$

where

$$a_k = ik\frac{1}{N}\sum_{j=0}^{N-1} u(x_j)e^{-ikx_j} = ik\tilde{u}_k.$$

Note that both these derivatives require both the forward and backward transform. Now, observe that

$$(\mathcal{D}_N u)_\ell = \sum_{j=0}^{N-1}\left(\underbrace{\left[\frac{1}{N}\sum_{k\in K_N} ike^{ik(x_\ell - x_j)}\right]}_{D_{\ell j}} u_j\right) \tag{7.3.7}$$

This can be realised as the matrix product $\mathcal{D}_N u = D\boldsymbol{u}$, where

$$D_{\ell j} = \begin{cases} \frac{1}{2}(-1)^{\ell+j}\cot\left[\frac{(\ell - j)\pi}{N}\right] & \text{for } \ell \neq j, \\ 0 & \text{for } \ell = j. \end{cases}$$

In summary,

$$\begin{aligned} \mathcal{D}_N u(x_\ell) &= \sum_{k\in K_N} ik\tilde{u}_k e^{ikx_\ell} = (I_N u)' \\ (P_N u)'(x_\ell) &= \sum_{k\in K_N} ik\hat{u}_k e^{ikx_\ell} = P_N(u') \end{aligned}$$

and they are not equal due to the aliasing error. For $u \notin S_N = \text{span}\{\phi_k, k \in K_N\}$, $(I_N u)' \neq I_N(u')$. It can be shown that that the quantity $D_N u - I_N u'$ depends in the same on k as does the quantity $u' - P_N u'$.

7.3.2 Viscous Burgers equation

Let us approximate u by

$$u^N(x,t) = \sum_{k\in K_N} \hat{u}_k(t)e^{ikx}.$$

In the Galerkin sense, we require

$$\big(u_t^N + u^N u_x^N - \beta u_{xx}^N, \phi_\ell\big) = 0 \quad \text{for } \ell \in K_N$$

and this is equivalent to

$$\frac{\partial \hat{u}_k}{\partial t} + \left(\widehat{u_N \frac{\partial u^N}{\partial x}}\right)_k + k^2\beta\hat{u}_k = 0 \quad \text{for } k \in k_N.$$

By definition,

$$\left(\widehat{u_N \frac{\partial u^N}{\partial x}}\right)_k = \frac{1}{2\pi}\int_0^{2\pi} u^N \frac{\partial u^N}{\partial x} e^{-ikx}\,dx. \tag{7.3.8}$$

In general, given $f(x), g(x)$, we want to find the Fourier transform of the product fg, *i.e.*

$$(\hat{fg})_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)g(x)e^{-ikx}\,dx.$$

For $f, g \in S_N$, we have that

$$f(x) = \sum_p \hat{f}_p e^{ipx}$$
$$g(x) = \sum_q \hat{g}_q e^{iqx}$$

and so

$$\begin{aligned}(\hat{fg})_k &= \sum_p \sum_q \hat{f}_p \hat{g}_q \frac{1}{2\pi}\int_0^{2\pi} e^{i(p+q-k)x}\,dx \\ &= \sum_{p+q=k} \hat{f}_p \hat{g}_q.\end{aligned}$$

This resemblance the property that the Fourier transform of the pointwise product fg is the convolution of the Fourier transforms of f and g.

Chapter 8

The Generalised Newtonian Fluid

In Chapter 2 we introduced the classical constitutive law for isotropic Newtonian fluid

$$\underline{\underline{T}} = \Big(-p + \lambda \nabla \cdot \boldsymbol{u} \Big) \underline{\underline{I}} + 2\mu \underline{\underline{E}},$$

where we assume that the stress depends linearly on the velocity gradient.

8.1 Newton's Law of Viscosity

Recall the stress tensor of an incompressible, isotropic Newtonian fluid:

$$\underline{\underline{T}} = -p\underline{\underline{I}} + 2\mu \underline{\underline{E}}, \tag{8.1.1}$$

where μ is the shear viscosity. The SI unit of the shear viscosity is $\mathrm{kgm^{-1}s^{-1}}$ or equivalently pascal second (Pa·s), Pa being the SI derived unit of pressure. To see this, we simply equate the dimension of terms in (8.1.1). Let M, L, T be the dimension of mass, length and time respectively. Then

$$\begin{aligned} [\underline{\underline{T}}] &= [\text{stress}] = \frac{[\text{force}]}{[\text{area}]} = \frac{ML}{T^2}\frac{1}{L^2} = \frac{M}{T^2 L} = [p] \\ [\underline{\underline{E}}] &= \frac{[\text{velocity}]}{[\text{time}]} = \frac{L}{T}\frac{1}{L} = \frac{1}{T} \end{aligned}$$

and so

$$[\mu] = \frac{[\underline{\underline{T}}]}{[\underline{\underline{E}}]} = \frac{M}{T^2 L} T = \frac{M}{LT}.$$

Consider two parallel rigid plates separated by a distance d, where the bottom plate is held stationary and the top plate is moving with velocity U. For sufficiently small U, the fluid layer near the top plate will move parallel to it by virtue of the no-slip condition and this motion induces the fluid layer just below it to move as well but with lower speed, due to the resisting force generated by the friction between adjacent layers. As such, an external force is needed to keep the top plate moving at constant speed. The magnitude F of this force is found to be proportional to the speed U and the area A of ech plate, and inversely proportional to their separation d

$$F \propto \frac{AU}{d}.$$

Fluids	Viscosity (Pa·s)
Air	10^{-5}
Water	10^{-3}
Ethyl Alcohol	1.2×10^{-3}
Mercury	1.5×10^{-3}
Ethylene	1.5×10^{-3}
Glycol	10^{-2}
Olive Oil	10^{-1}
Glycerol	1.5
Honey	10
Corn Syrup	100

Figure 8.1: Some values of viscosity of fluids, in pascal second, measured at room temperature.

This is known as **Newton's Law of Viscosity** and the constant of proportionality is called the shear viscosity of the fluid. The ratio $\dot{\gamma} = U/d$ is known as the *rate of shear strain* or *shear velocity* and it is the derivative of the fluid speed in the direction perpendicular to the plates, *i.e.* the velocity gradient. In terms of the shear stress $\tau = F/A$, we have the following simple equation

$$\tau = \mu \frac{\partial u}{\partial y} = \mu \dot{\gamma}. \tag{8.1.2}$$

This linear relationship between τ and $\dot{\gamma}$ is a distinct feature of a Newtonian fluid, and fluids that do not obey such stress-shear rate are called non-Newtonian fluids. Examples of non-Newtonian fluids are:

1. biological fluids such as blood, synovial fluid and mucus;
2. cosmetics such as lotions, shaving creams and nail polish;
3. food such as chocolate, yogurt, peanut butter and mayonaise;
4. fire fighting foams, lubricating oils, magma, sludge.

8.2 Non-Newtonian Viscosity

Plotting the stress versus the shear rate $\dot{\gamma}$, we obtain a line in which the viscosity μ is simply the slope of this line;

Definition 8.2.1. A generalised Newtonian fluid (GNF) is a fluid for which the value of $\dot{\gamma}$ at a point is determined by the current state, *i.e.* it has no memory and independent of time.

Definition 8.2.2. A power law fluid is a fluid such that the following relation holds:

$$\underline{\underline{\sigma}} = \kappa \dot{\gamma}^n,$$

where σ is the stress. Such equation is known as Ostwald de Waede equation. Note that if $0 < n < 1$, then the fluid is shear thinning; in practice $n \approx 0.3 - 0.7$.

In general, for a non-Newtonian fluid,

$$\underline{\underline{T}} = -p\underline{\underline{I}} + \text{ some other terms, which we will call } \sigma.$$

Note that $\operatorname{tr}\left(\underline{\underline{\sigma}}\right) = 0$ if the fluid is Newtonian and in general, we can always define $\underline{\underline{\sigma}}$ to be traceless by absorbing the trace into the pressure term.

Example 8.2.3. Consider the pipe flow on a cylindrical pipe.

8.3 Pipe Flow

Consider a power law fluid in an axisymmetric cylinder. Assuming unidirectional flow in the z-direction, we have $u_z(R) = 0$. The PDE is

$$0 = -\frac{\Delta p}{L} + \frac{1}{r}\partial_r\left(r\sigma_{zr}\right),$$

with boundary condition $\sigma_{zr}(R) = \sigma_w$. The general solution is

$$\sigma_{zr} = \frac{r}{2}\frac{\Delta P}{L} + \frac{C}{r}.$$

We want bounded solutions at $r = 0$, which implies $C = 0$. Thus,

$$\sigma_{zr} = \frac{r}{2}\frac{\Delta P}{L}.$$

Finally, applying the boundary condition, we obtain an expression for σ_{zr} in terms of σ_w:

$$\sigma_{zr} = \frac{r}{R}\sigma_w.$$

The rate of strain tensor in cylindrical coordinate only has one non-zero component: $E_{zr} = E_{rz}$. So $2E_{zr} = \partial_r u_z = \dot{\gamma}$. In this problem, $\dot{\gamma} < 0$ since the velocity decreases away from the center line, and so

$$\kappa\left|\partial_r u_z\right|^n = \frac{r}{R}\sigma_w \implies -\partial_r u_z = \left(\frac{r}{R}\sigma_w\right)^{1/n}.$$

8.3.1 Coutte Flow

8.3.2 General Strategy for Solving Isothermal Flow Problems

1. Physics - always five

   - Conservation of mass $\frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{u} = 0$;
   - Conservation of linear momentum $\rho\left(\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = \boldsymbol{F}_b + \nabla\cdot\underline{\underline{T}}$;
   - Conservation of angular momentum $\underline{\underline{T}} = \underline{\underline{T}}^T$.

   Problem dependent incompressible flow ρ constant: $\nabla\cdot\boldsymbol{u} = 0$.

2. Geometry - BC/IC, problem dependent. Geometry tells us which coordinates system to use: Cartesian, cylindrical or spherical for instance.

   BC: the most common ones are no-slip and no-penetration, usually given for $\boldsymbol{u}$.

3. Constitutive equation for $\underline{\underline{T}}$: It relates $\underline{\underline{T}}$ to some of the other unknowns, in particular to $\nabla \boldsymbol{u}$. Note that we can have $\underline{\underline{T}} = -p\underline{\underline{I}} + \underline{\underline{\sigma}}$ with $\operatorname{tr}\left(\underline{\underline{\sigma}}\right) = 0$, where p might be some other quantity other than the hydrostatic pressure (think of it as the Lagrange multiplier for the incompressibility constraint); we then want to relate $\underline{\underline{\sigma}}$ to some other unknowns. We want to express $\underline{\underline{\sigma}}$ as a function of $\underline{\underline{E}}$ and not $\nabla \boldsymbol{u}$ or $\underline{\underline{E}}$ and $\underline{\underline{W}}$ (anti-symmetric part of $\nabla \boldsymbol{u}$),*i.e.*
$$\underline{\underline{\sigma}} = f\left(\underline{\underline{E}}, \text{ higher-order derivatives of } \underline{\underline{E}}\right).$$

All of this generates a PDE problem that is solvable in theory, but it can be difficult to solve by hand! Let's discuss a little bit further about f. If we have a Newtonian fluid, then $\underline{\underline{\sigma}} = 2\mu\underline{\underline{E}}$ for an incompressible flow, *i.e.* f is linear in $\underline{\underline{E}}$. For Bingham fluid, $\underline{\underline{\sigma}} = \underline{\underline{\sigma_y}} + 2\mu\underline{\underline{E}}$. Note that implicitly we are assuming there is no time memory in the fluid, which means that the current state $\underline{\underline{E}}(T)$ only depends on T but not on previous times $t < T$. In the special case of the steady unidirectional flow, the system of PDEs reduces to a simpler system; note that the direction of the flow depends on the geometry and also the chosen coordinate system. The Navier-Stokes equation becomes
$$0 = -\nabla p + \nabla \cdot \underline{\underline{\sigma}},$$
together with the incompressibility condition $\nabla \cdot \boldsymbol{u} = 0$ and the constitutive relation $\underline{\underline{\sigma}} = f\left(\underline{\underline{E}}\right)$.

For a 1D unidirectional flow, there is only one (two) non-zero component of $\underline{\underline{E}}$. For example, if $\boldsymbol{u} = u(y)\boldsymbol{e}_1$, then $2E_{xy} = 2E_{yx} = \partial_y u$; if $\boldsymbol{v} = v(r)\boldsymbol{e}_z$, then $2E_{rz} = 2E_{zr} \neq 0$. This implies that $\underline{\underline{\sigma}}$ has only one (two) non-zero component(s). In other words, we have a scalar constitutive law:
$$\sigma = f(e), \quad \text{where } e \text{ is the corresponding nonzero component of } \underline{\underline{E}}.$$

This now greatly simplifies the vector equations (3 components); in fact, $\underline{\underline{\sigma}}$ only shows up in 1 equation while the other equations only involve p. As a result, we obtain a first order ODE for $\underline{\underline{\sigma}}$ which may depend on p, which we can solve it explicitly most of the time. There will be an integration constant C and in general we do not want to solve for C unless we can use the fact σ should not blow up. Finally, we plug σ into the scalar constitutive equation, which will give a first order ODE for u because e is some derivative of u. We can solve it and use the BC on $\boldsymbol{u}$ to eliminate the constant of integration D. (Note that now we have two integration constants.)

Chapter 9

Linear Viscoelasticity

The goal is to write a constitutive equation for the stress. We look for fluid/solid that behaves like a fluid in some regime and like a solid in others.

9.1 Maxwell (Viscoelastic) Fluid

9.2 Creep Test

We want to generalise the one-dimensional Maxwell model to a three-dimensional model, *i.e.* tensors. We exploit the principle of frame invariance. Roughly speaking, a constitutive law should not change under rigid body rotation.

Definition 9.2.1. Let $\underline{\underline{F}} = \frac{d\boldsymbol{\alpha}}{d\boldsymbol{x}}$, then $\underline{\underline{F}}^{-1}$ is the deformation gradient.

blablabla

The correct quantity to use is the Finger tensor, defined by

$$\underline{\underline{C}}^{-1} = \left(\underline{\underline{F}}^{-1}\right)^T \underline{\underline{F}}.$$

In fact, the correct constitutive relationship for linear elasticity is $\underline{\underline{\sigma}} = G\underline{\underline{C}}^{-1}$.

Chapter 10

Numerical Solutions to Navier-Stokes Equations

We nondimensionalize the dimensional incompressible Navier Stokes equations with dimensionless variables

$$\tilde{\boldsymbol{x}} = L\boldsymbol{x}, \quad \tilde{\boldsymbol{u}} = U\boldsymbol{u}, \quad \tilde{t} = \frac{L}{U}t, \quad \tilde{p} = \rho U^2 p, \quad \tilde{\boldsymbol{f}} = \frac{\rho U^2}{L}\boldsymbol{f},$$

and define the Reynolds number

$$\mathrm{Re} = \frac{\rho U L}{\mu}.$$

The dimensionless incompressible Navier Stokes equations in the domain Ω are

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla \boldsymbol{u} + \boldsymbol{f} \tag{10.0.1a}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{10.0.1b}$$

We impose the boundary condition

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}_{\text{boundary}}(\boldsymbol{x}, t) \quad \text{for } x \in \partial\Omega,$$

and the initial condition

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}).$$

10.1 Projection Methods

These methods are motivated by a decomposition theorem for vector fields:

Theorem 10.1.1 (Hodge Decomposition)**.** *Let $\boldsymbol{w}(\boldsymbol{x})$ be a smooth vector field defined on a region Ω. Then $\boldsymbol{w}(\boldsymbol{x})$ can be written as*

$$\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x}) + \nabla\phi(\boldsymbol{x}),$$

where $\boldsymbol{v}$ is divergence-free and $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$.

Proof. If this were true, then ϕ satisfies the Poisson equation with Neumann boundary condition since

$$\begin{aligned} \nabla \cdot \boldsymbol{w} = \nabla \cdot (\nabla \phi) = \Delta \phi & \quad \text{in } \Omega & \qquad (10.1.1a) \\ \nabla \phi \cdot \boldsymbol{n} = \boldsymbol{w} \cdot \boldsymbol{n} & \quad \text{on } \partial\Omega & \qquad (10.1.1b) \end{aligned}$$

With this in mind, we define $\phi(\boldsymbol{x})$ to be the solution of (10.1.1). For (10.1.1) to have a solution, a compatibility condition must hold, which can be found by integrating by parts:

$$\int_\Omega f \, dV_{\boldsymbol{x}} = \int_\Omega \Delta \phi \, dV_{\boldsymbol{x}} = \int_{\partial\Omega} \nabla \phi \cdot \boldsymbol{n} \, dS_{\boldsymbol{x}} = \int_{\partial\Omega} g \, dS_x.$$

By assumption, this compatibility condition holds and (10.1.1) has a unique solution $\phi(\boldsymbol{x})$ up to additive constant. By setting $\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{w}(\boldsymbol{x}) - \nabla \phi(\boldsymbol{x})$, one can show that it satisfies the required constraint and the theorem is proved. ■

Remark 10.1.2. Note that the Hodge decomposition is unique. Indeed,

$$\begin{aligned} \int_\Omega \boldsymbol{v} \cdot \nabla \phi \, dV_{\boldsymbol{x}} &= \int_\Omega \nabla \cdot (\phi \boldsymbol{v}) - \phi \nabla \cdot \boldsymbol{v} \, dV_{\boldsymbol{x}} \\ &= \int_\Omega \nabla \cdot (\phi \boldsymbol{v}) \; dV_{\boldsymbol{x}} \\ &= \int_{\partial\Omega} \phi \boldsymbol{v} \cdot \boldsymbol{n} \, dS_{\boldsymbol{x}} = 0, \end{aligned}$$

i.e. $\boldsymbol{v}$ and $\nabla \phi$ are orthogonal in $L^2(\Omega)$. This also means that given a vector field $\boldsymbol{w}$, we have that $\boldsymbol{v} = P(\boldsymbol{w})$, where P is the linear orthogonal projection operator (onto where?). CHT:Explain this decomposition better, see Kopachevsky book.

If $\nabla \cdot \boldsymbol{u}(\boldsymbol{x}, t) = 0$ for all $t \geq 0$, then $\nabla \cdot \boldsymbol{u}_t(\boldsymbol{x}, t) = 0$. Rearranging the (10.0.1), we have that

$$\boldsymbol{u}_t + \nabla p = -\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{1}{\text{Re}} \Delta \boldsymbol{u} + \boldsymbol{f} = \boldsymbol{w}, \qquad (10.1.2)$$

and the LHS is a decomposition of some vector field. From above, if $\nabla \cdot \boldsymbol{w} = 0$, then $\boldsymbol{w} = P(\boldsymbol{w})$; if $\boldsymbol{w} = \nabla \phi$, then $P(\boldsymbol{w}) = 0$. Applying P onto (10.1.2), we obtain

$$\boldsymbol{u}_t = P\left(-\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{1}{\text{Re}} \Delta \boldsymbol{u} + \boldsymbol{f} \right).$$

We discretise time into steps of size Δt. Let $t_n = n \Delta t$. Then

$$\begin{aligned} \frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} + \boldsymbol{A}^{n+\frac{1}{2}} &= -\nabla p^{n+\frac{1}{2}} + \frac{1}{2\text{Re}} \left(\nabla \boldsymbol{u}^{n+1} + \nabla \boldsymbol{u}^n \right) + \boldsymbol{f}^{n+\frac{1}{2}} & \qquad (10.1.3) \\ \nabla \cdot \boldsymbol{u}^{n+1} &= 0, & \qquad (10.1.4) \end{aligned}$$

where $\boldsymbol{f}^{n+\frac{1}{2}}$ is the vector $\boldsymbol{f}$ evaluated at $\left(n + \frac{1}{2}\right) \Delta t$ and $A^{n+\frac{1}{2}}$ is some second-order in time approximation to $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ at $\left(n + \frac{1}{2}\right) \Delta t$.

Chapter 11

Continuum Theory of Polymeric Fluids at Equilibrium

[**morozov2015introduction**]

11.1 Mechanical Models for Polymer Molecules

We consider flexible polymers. They can take up on enormous number of configurations from the rotation of chemical bond.

11.1.1 Freely jointed bead-rod chain model

Consider a freely jointed chain of $(N+1)$ beads with position vectors $\{\boldsymbol{R}_0, \ldots, \boldsymbol{R}_N\}$ and N bonds or links $\boldsymbol{R}_i = \boldsymbol{R}_i - \boldsymbol{R}_{i-1}, i = 1, 2, \ldots, N$ with bond length b_0. Let $\psi(\boldsymbol{R})$ be the distribution function of a random vector of length b_0, in the sense that the polar angles for the vector in the chain are completely random. Denote the following:

$$\{\boldsymbol{R}_n\} = (\boldsymbol{R}_0, \ldots, \boldsymbol{R}_N\mathbb{R}), \quad \{\boldsymbol{R}_n\} = (\boldsymbol{R}_1, \boldsymbol{R}_2, \ldots, \mathbb{R}).$$

Since each bond is assumed to be oriented independently of all other bonds, the equilibrium polymer conformation/configuration distribution function for the entire chain is simply the product of the single bond distribution functions:

$$\Psi\left(\{\boldsymbol{R}_n\}\mathbb{R}\right) = \prod_{n=1}^{N} \psi(\boldsymbol{R}_n).$$

We have

$$\psi(\boldsymbol{R}) = \frac{1}{4\pi b_0^2}\delta\left(|\boldsymbol{R}| - b_0\mathbb{R}\right), \quad \text{such that} \quad \int_{\boldsymbol{R}^3} \psi(\boldsymbol{R})\, d\boldsymbol{R} = 1,$$

i.e. $\psi(\boldsymbol{R})$ is a probability density function.

Let $\boldsymbol{R}$ be the *end-to-end displacement vector* defined by

$$\boldsymbol{R} = \boldsymbol{R}_N - \boldsymbol{R}_0 = \sum_{n=1}^{N} \boldsymbol{R}_n - \boldsymbol{R}_{n-1} = \sum_{n=1}^{N} \boldsymbol{R}_n.$$

Then

$$\langle \boldsymbol{R} \rangle = \sum_{n=1}^{N} \langle \boldsymbol{R}_n \rangle = 0$$

$$\langle \boldsymbol{R}^2 \rangle = \langle \boldsymbol{R} \cdot \boldsymbol{R} \rangle = \left\langle \sum_{n=1}^{N} \boldsymbol{R}_n \cdot \sum_{m=1}^{N} \boldsymbol{R}_m \mathbb{R} \right\rangle = \sum_{n,m=1}^{N} \langle \boldsymbol{R}_n \cdot \boldsymbol{R}_m \rangle = \sum_{n=1}^{N} \langle \boldsymbol{R}_n \rangle^2 + 2 \sum_{\substack{n,m=1 \\ n>m}}^{N} \langle \boldsymbol{R}_n \cdot \boldsymbol{R}_m \rangle = \sum_{n=1}^{N} \langle \boldsymbol{R}_n \rangle^2$$

We use spherical coordinate to compute $\langle \boldsymbol{R}^2 \rangle$:

$$\begin{aligned}
\langle \boldsymbol{R}^2 \rangle = \int_{\boldsymbol{R}^3} |\boldsymbol{R}|^2 \psi(\boldsymbol{R}) \, d\boldsymbol{R} &= \frac{1}{4\pi b_0^2} \int_{\boldsymbol{R}^3} |\boldsymbol{R}|^2 \delta\left(|r| - b_0 \mathbb{R}\right) \, d\boldsymbol{R} \\
&= \frac{1}{4\pi b_0^2} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \boldsymbol{Rr}\rho^2 \delta\left(\boldsymbol{Rr}\rho - b_0 \mathbb{R}\right) \boldsymbol{Rr}\rho^2 \sin\theta \, d\boldsymbol{R}\rho \, d\theta \, d\phi \\
&= \frac{1}{b_0^2} \int_0^{\infty} \boldsymbol{Rr}\rho^4 \delta\left(\boldsymbol{Rr}\rho - b_0 \mathbb{R}\right) \, d\boldsymbol{Rr}\rho \\
&= \frac{b_0^4}{b_0^2} = b_0^2.
\end{aligned}$$

Hence, $\langle \boldsymbol{R}^2 \rangle = N b_0^2$ and the *root mean square* of the end-to-end displacement vector is

$$\overline{\boldsymbol{R}} = \sqrt{\langle \boldsymbol{R}^2 \rangle} = \sqrt{N} b_0.$$

Remark 11.1.1. The fact that $\langle \boldsymbol{R}^2 \rangle \sim N$ holds for general model, *e.g.* freely rotating chain, in the limit that N is large. In general we have $\langle \boldsymbol{R}^2 \rangle = N b^2$, where b is the effective bond length, which can be computed from the stiffness $c_\infty = b^2 / b_0^2$. For example,

$$\langle \boldsymbol{R}^2 \rangle = N b_0^2 \left(\frac{1 + \cos\theta}{1 - \cos\theta} \mathbb{R} \right) \quad \text{for large } N.$$

CHT:I don't know what is the b_k term on your note.

11.1.2 Distribution of the end-to-end displacement vector

Let $\phi(\boldsymbol{R}, N)$ be the probability distribution function that the end-to-end displacement vector of the chain consisting of N links of length b is $\boldsymbol{R}$. The probability of such event is given by

$$\Phi(\boldsymbol{R}, N) = \int d\boldsymbol{R}_1 \int d\boldsymbol{R}_2 \cdots \int d\boldsymbol{R}_N \delta \left(\boldsymbol{R} - \sum_{n=1}^{N} \boldsymbol{R}_n \mathbb{R} \right) \Psi\Big(\{\boldsymbol{R}_n\} \Big).$$

Recall the Fourier transform and inverse Fourier transform:

$$\begin{aligned}
\hat{f}(\boldsymbol{k}) &= \int_{\boldsymbol{R}^3} f(\boldsymbol{x}) e^{-i\boldsymbol{x}\cdot\boldsymbol{k}} \, d\boldsymbol{x} \\
f(\boldsymbol{x}) &= \frac{1}{(2\pi)^3} \int_{\boldsymbol{R}^3} \hat{f}(\boldsymbol{k}) e^{i\boldsymbol{x}\cdot\boldsymbol{k}} \, d\boldsymbol{k}.
\end{aligned}$$

One can show that

$$\hat{\delta}(\boldsymbol{k}) = 1 \implies \delta(\boldsymbol{R}) = \frac{1}{(2\pi)^3}\int_{\boldsymbol{R}^3} e^{i\boldsymbol{R}\cdot\boldsymbol{k}}\, d\boldsymbol{k}.$$

Consequently,

$$\begin{aligned}
\Phi(\boldsymbol{R}, N) &= \frac{1}{(2\pi)^3}\int d\boldsymbol{R}_1 \int d\boldsymbol{R}_2 \cdots \int d\boldsymbol{R}_N \int d\boldsymbol{k} \exp\left(i\boldsymbol{k}\cdot\left(\boldsymbol{R} - \sum_{n=1}^{N}\boldsymbol{R}_n\mathbb{R}\right)\mathbb{R}\right)\Psi\Big(\{\boldsymbol{R}_n\}\Big) \\
&= \frac{1}{(2\pi)^3}\int d\boldsymbol{R}_1 \int d\boldsymbol{R}_2 \cdots \int d\boldsymbol{R}_N \int d\boldsymbol{k} \exp\left(i\boldsymbol{k}\cdot\boldsymbol{R}\mathbb{R}\right)\exp\left(-i\boldsymbol{k}\cdot\sum_{n=1}^{N}\boldsymbol{R}_n\mathbb{R}\right)\Psi\Big(\{\boldsymbol{R}_n\}\Big) \\
&= \frac{1}{(2\pi)^3}\int d\boldsymbol{k} \exp\left(i\boldsymbol{k}\cdot\boldsymbol{R}\mathbb{R}\right)\int d\boldsymbol{R}_1 \int d\boldsymbol{R}_2 \cdots \int d\boldsymbol{R}_N \prod_{n=1}^{N}\exp\left(-i\boldsymbol{k}\cdot\boldsymbol{R}_n\mathbb{R}\right)\psi(\boldsymbol{R}_n) \\
&= \frac{1}{(2\pi)^3}\int d\boldsymbol{k} \exp\left(i\boldsymbol{k}\cdot\boldsymbol{R}\mathbb{R}\right)\left[\int d\boldsymbol{R} \exp\left(-i\boldsymbol{k}\cdot\boldsymbol{R}\mathbb{R}\right)\psi(\boldsymbol{R})\mathbb{R}\right]^N.
\end{aligned}$$

Substituting the bond distribution function $\psi(\boldsymbol{R})$ gives

$$\begin{aligned}
\int_{\boldsymbol{R}^3}\exp\left(-i\boldsymbol{k}\cdot\boldsymbol{R}\mathbb{R}\right)\psi(\boldsymbol{R})\, d\boldsymbol{R} &= \frac{1}{4\pi b^2}\int_{\boldsymbol{R}^3}\exp\left(-i\boldsymbol{k}\cdot\boldsymbol{R}\mathbb{R}\right)\delta\left(|\boldsymbol{R}| - b\mathbb{R}\right)\, d\boldsymbol{R} \\
&= \frac{1}{4\pi b^2}\int_{\boldsymbol{R}^3}\exp\left(-i|\boldsymbol{k}||\boldsymbol{R}|\cos\theta\mathbb{R}\right)\delta\left(|\boldsymbol{R}| - b\mathbb{R}\right)\, d\boldsymbol{R},
\end{aligned}$$

where θ is the angle between $\boldsymbol{R}$ and $\boldsymbol{k}$. Because the integrand is invariant under rotations, we can employ a modified spherical coordinates $(\boldsymbol{R}\rho, \phi, \theta)$ such that θ is the angle between $\boldsymbol{R}$ and the z-axis. Denote $k = |\boldsymbol{k}|$, this results in

$$\begin{aligned}
&\frac{1}{4\pi b^2}\int_{\boldsymbol{R}^3}\exp\left(-i|\boldsymbol{k}||\boldsymbol{R}|\cos\theta\mathbb{R}\right)\delta\left(|\boldsymbol{R}| - b\mathbb{R}\right)\, d\boldsymbol{R} \\
&= \frac{1}{4\pi b^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\exp\left(-ik\boldsymbol{R}\rho\cos\theta\mathbb{R}\right)\delta(\boldsymbol{R}\rho - b)\boldsymbol{R}\rho^2\sin\theta\, d\boldsymbol{R}\rho\, d\theta\, d\phi \\
&= \frac{2\pi}{4\pi b^2}\int_0^{\pi} b^2\exp\left(-ikb\cos\theta\mathbb{R}\right)\sin\theta\, d\theta \\
&= \frac{1}{2}\int_{-1}^{1} e^{-ikbu}\, du \\
&= -\frac{1}{2ikb}e^{-ikbu}\Big|_{-1}^{1} \\
&= -\frac{1}{2ikb}\left(e^{-ikb} - e^{ikb}\mathbb{R}\right) \\
&= \frac{\sin(kb)}{kb},
\end{aligned}$$

where we make a change of variable $u = \cos\theta$.

For $kb \ll 1$,

$$\sin(kb) \approx kb - \frac{(kb)^3}{3!}$$

$$\frac{\sin(kb)}{kb} \approx 1 - \frac{k^2b^2}{6}$$
$$\left(\frac{\sin(kb)}{kb}\mathbb{R}\right)^N \approx \left(1 - \frac{k^2b^2}{6}\mathbb{R}\right)^N = \left(1 - \frac{k^2b^2N}{6N}\mathbb{R}\right)^N \longrightarrow e^{-k^2b^2N/6} \quad \text{as } N \longrightarrow \infty.$$

Hence, for sufficiently small kb and sufficiently large N,

$$\left(\frac{\sin(kb)}{kb}\mathbb{R}\right)^N \approx e^{-k^2b^2N/6}.$$

CHT:Note that the absolute max of $\sin(kb)/kb$ is at $k = 0$ and one can use asymptotic expansion of integrals to prove it? Thus we have

$$\begin{aligned}
\Phi(\boldsymbol{R}, N) &= \frac{1}{(2\pi)^3}\int_{\boldsymbol{R}^3} e^{i\boldsymbol{k}\cdot\boldsymbol{R}} e^{-k^2b^2N/6}\, d\boldsymbol{k} \\
&= \frac{1}{(2\pi)^3}\int_{\boldsymbol{R}^3} e^{i(k_xR_x + k_yR_y + k_zR_z\mathbb{R})} e^{-b^2N\left(k_x^2+k_y^2+k_z^2\mathbb{R}\right)/6}\, dk_x\, dk_y\, dk_z \\
&= \frac{1}{(2\pi)^3}\prod_{\alpha=1}^{3}\int_{\boldsymbol{R}} e^{ik_\alpha R_\alpha} e^{-b^2Nk_\alpha/6}\, dk_\alpha \\
&= \frac{1}{(2\pi)^3}\prod_{\alpha=1}^{3}\sqrt{\frac{6\pi}{Nb^2}}\exp\left(-\frac{3}{2Nb^2}R_\alpha^2\mathbb{R}\right) \\
&= \frac{1}{(4\pi^2)^{3/2}}\left(\frac{6\pi}{Nb^2}\mathbb{R}\right)^{3/2}\exp\left(-\frac{3}{2Nb^2}|\boldsymbol{R}|^2\mathbb{R}\right) \\
&= \left(\frac{3}{2\pi Nb^2}\mathbb{R}\right)^{3/2}\exp\left(-\frac{3|R|^2}{2Nb^2}\mathbb{R}\right),
\end{aligned}$$

i.e. $\Phi(\boldsymbol{R}, N)$ is Gaussian. Note that this approximation is bad if $|\boldsymbol{R}|^2$ is larger than Nb, the maximum extended length of the chain. This is true in general as long as

$$\Psi\left(\{\boldsymbol{R}_n\}\mathbb{R}\right) = \prod_n \psi\left(\boldsymbol{R}_n, \boldsymbol{R}_{n+1}, \ldots, \boldsymbol{R}_{n+\boldsymbol{n}_c}\mathbb{R}\right),$$

by Central Limit Theorem.

11.1.3 Gaussian chain

Consider a chain whose bond length is Gaussian distributed as follows:

$$\psi(\boldsymbol{R}) = \left(\frac{3}{2\pi b^2}\mathbb{R}\right)^{3/2}\exp\left(-\frac{3|\boldsymbol{R}|^2}{2b^2}\mathbb{R}\right), \quad \text{with } \langle \boldsymbol{R}^2\rangle = b^2.$$

The configuration distribution distribution function is given by

$$\Psi\left(\{\boldsymbol{R}_n\}\mathbb{R}\right) = \prod_{n=1}^{N}\left(\frac{3}{2\pi b^2}\mathbb{R}\right)^{3/2}\exp\left(-\frac{3|\boldsymbol{R}_n|^2}{2b^2}\mathbb{R}\right) = \left(\frac{3}{2\pi b^2}\mathbb{R}\right)^{3N/2}\exp\left(-\sum_{n=1}^{N}\frac{3\left|\boldsymbol{R}_n - \boldsymbol{R}_{n-1}\mathbb{R}\right|^2}{2b^2}\mathbb{R}\right)$$

Instead of viewing the beads joining by rigid bonds with Gaussian distributed length, we represent this with a mechanical model where the beads are now connected by harmonic springs with potential energy

$$U_0\Big(\{\boldsymbol{R}_n\}\Big) = \frac{H}{2}\sum_{n=1}^{N} |\boldsymbol{R}_n - \boldsymbol{R}_{n-1}\mathbb{R}|^2,$$

where H is the spring constant. In the case of Gaussian distribution, the spring constant has a simple expression

$$H = \frac{3k_BT}{b^2},$$

where T is the temperature at equilibrium and k_B is the Boltzmann constant. Note that at equilibrium the Maxwell-Boltzmann distribution is exactly $\Psi(\{\boldsymbol{R}_n\})$.

A similar calculation in Section 5.2 leads to the following:

$$\Phi\left(\boldsymbol{R}_n - \boldsymbol{R}_m, n - m\mathbb{R}\right) = \left(\frac{3}{2\pi b^2|n-m|}\mathbb{R}\right)^{3/2} \exp\left(-\frac{3|\boldsymbol{R}_n - \boldsymbol{R}_m|^2}{2|n-m|b^2}\mathbb{R}\right),$$

which implies that $\langle|\boldsymbol{R}_n - \boldsymbol{R}_m|^2\rangle = |n-m|b^2$. Later on, we are going to continuous variable, *i.e.* $\boldsymbol{R}_n - \boldsymbol{R}_{n-1} \longrightarrow \dfrac{\partial \boldsymbol{R}_n}{\partial \boldsymbol{R}}$; this leads to the Wiener distribution

$$\Psi\left(\boldsymbol{R}_n\mathbb{R}\right) = C\exp\left(-\frac{3}{2b^2}\int_0^N \left|\frac{\partial \boldsymbol{R}_n}{\partial \boldsymbol{R}}\mathbb{R}\right|^2 d\boldsymbol{R}\mathbb{R}\right).$$

Remark 11.1.2. If the springs are taken to be Hookean springs, then the freely jointed bead-spring chain is called a **Rouse-Zimm chain**. It contains three parameters: the number of beads N, the Hookean spring constant H and the Stokes' drag coefficient ξ.

11.1.4 Dumbbell models

The special case of $N = 2$ is referred to as the dumbbell models. The spring force is $\boldsymbol{F}^c = \dfrac{\partial}{\partial \boldsymbol{R}}\phi^c$, where ϕ^c is the spring potential energy.

1. **Hookean (linear)**: $\phi^c = \dfrac{1}{2}H|\boldsymbol{R}|^2$. Hookean springs are infinitely stretchable.

2. **Fraenkel**: $\phi^c = \dfrac{1}{2}\left(|\boldsymbol{R}| - |\boldsymbol{R}_0|\mathbb{R}\right)^2$, where $|\boldsymbol{R}_0|$ is the preferred rest length. We recover Hookean springs when $|r_0| \longrightarrow 0$ and a rigid rod of length $|\boldsymbol{R}_0|$ when $H \longrightarrow \infty$.

3. **Tanner (linear-locked)**: $\phi^c = \dfrac{1}{2}H|\boldsymbol{R}|^2$ if $|\boldsymbol{R}| < |\boldsymbol{R}_0|$. These springs can stretch only as far as $|\boldsymbol{R}| = |\boldsymbol{R}_0|$.

4. **Warner (FENE)**: $\phi^c = -\dfrac{1}{2}H|\boldsymbol{R}_0|^2\ln\left(1 - \left|\dfrac{\boldsymbol{R}}{\boldsymbol{R}_0}\mathbb{R}\right|^2\mathbb{R}\right)$ if $|\boldsymbol{R}| < |\boldsymbol{R}_0|$. These "finite extendable nonlinear elastic" (FENE) springs have an upper limiting length of $|\boldsymbol{R}| = |\boldsymbol{R}_0|$.

11.2 Elastic Dumbbell Models

In this section we study the elastic dumbbell model. In equilibrium systems one can write down a formal expression for the configurational distribution function directly by means of equilibrium statistical mechanics. For nonequilibrium systems the configurational distribution function satisfies a second-order PDE and this can be solved analytically only for simple macromolecular models. We also derive an expression for the stress tensor.

11.2.1 Modeling assumptions

We treat the polymer molecule as an elastic dumbbell, that is we have two beads of mass m, labelled ① and ②, connected by a spring. Their spatial locations are denoted by $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$. It is sometimes convenient to work with the "connector vector" $\boldsymbol{Q} = \boldsymbol{R}_2 - \boldsymbol{R}_1$ and the center of mass $\boldsymbol{R}_c = (\boldsymbol{R}_1 + \boldsymbol{R}_2)/2$. $\boldsymbol{Q}$ describes the overall orientation and the internal configuration of the polymer molecule. Most kinetic theories have the following assumptions:

1. The flow field of the polymer solution is homogeneous, in the sense that the rate of strain tensor $\underline{\underline{E}}$ is the same everywhere in the flow field. Therefore the mass-average velocity field $\boldsymbol{v}$ is linear, *i.e.*

$$\boldsymbol{v} = \boldsymbol{v}_0 + \underline{\underline{\kappa}} \cdot \boldsymbol{R}, \tag{11.2.1}$$

   for some $\boldsymbol{v}_0$ independent of $\boldsymbol{R}$. Because of the incompressibility condition, $\underline{\underline{\kappa}}$ is a traceless tensor.

2. The phase space distribution function $f(\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, t)$ is replaced by the corresponding distribution function $F(\boldsymbol{R}_1, \boldsymbol{R}_2, \dot{\boldsymbol{R}}_1, \dot{\boldsymbol{R}}_2, t)$ in the position-velocity space, such that

$$F(\boldsymbol{R}_1, \boldsymbol{R}_2, \dot{\boldsymbol{R}}_1, \dot{\boldsymbol{R}}_2, t) = \Psi(\boldsymbol{R}_1, \boldsymbol{R}_2, t)\Xi(\dot{\boldsymbol{R}}_1, \dot{\boldsymbol{R}}_2, \boldsymbol{R}_1, \boldsymbol{R}_2, t), \tag{11.2.2}$$

   where

$$\begin{aligned} \Psi(\boldsymbol{R}_1, \boldsymbol{R}_2, t) &= \text{configuration-space distribution function} \\ \Xi(\dot{\boldsymbol{R}}_1, \dot{\boldsymbol{R}}_2, \boldsymbol{R}_1, \boldsymbol{R}_2, t) &= \text{velocity-space distribution function.} \end{aligned}$$

   We impose the normalization condition

$$\iint \Xi \, d\dot{\boldsymbol{R}}_1 \, d\dot{\boldsymbol{R}}_2 = 1.$$

3. The configuration-space distribution function Ψ can be factored as

$$\Psi(\boldsymbol{R}_1, \boldsymbol{R}_2, t) = n\psi(\boldsymbol{Q}, t), \quad \int \psi(\boldsymbol{Q}, t) \, d\boldsymbol{Q} = 1, \tag{11.2.3}$$

   where n is the number of polymer molecules per unit volume. This indicates that the configuration distribution is independent of the location of the dumbbells in space (or the center of mass).

4. The velocity-space distribution function Ξ is Maxwellian about the mass-average solution velocity $\boldsymbol{v}$ at the center of mass of the dumbbell, that is

$$\Xi_{\text{eq}}(\dot{\boldsymbol{R}}_1, \dot{\boldsymbol{R}}_2) = \frac{\exp\left(-\frac{1}{kT}\left[\frac{1}{2}m|\dot{\boldsymbol{R}}_1 - \boldsymbol{v}|^2 + \frac{1}{2}m|\dot{\boldsymbol{R}}_2 - \boldsymbol{v}|^2\mathbb{R}\right]\mathbb{R}\right)}{\iint_{\boldsymbol{R}^2}\exp\left(-\frac{1}{kT}\left[\frac{1}{2}m|\dot{\boldsymbol{R}}_1 - \boldsymbol{v}|^2 + \frac{1}{2}m|\dot{\boldsymbol{R}}_2 - \boldsymbol{v}|^2\mathbb{R}\right]\mathbb{R}\right)d\dot{\boldsymbol{R}}_1\,d\dot{\boldsymbol{R}}_2}. \tag{11.2.4}$$

In other words, the velocity distribution is the same as that in a solution at equilibrium. This is sometimes called the assumption of "equilibration in momentum space".

Before we explain the four kinds of forces experienced by each bead, we introduce the following two notations. Given any time-independent function $B(\boldsymbol{R}_1, \boldsymbol{R}_2, \dot{\boldsymbol{R}}_1, \dot{\boldsymbol{R}}_2)$, its velocity-space average is given by

$$[\![B]\!] = \iint B(\boldsymbol{R}_1, \boldsymbol{R}_2, \dot{\boldsymbol{R}}_1, \dot{\boldsymbol{R}}_2)\Xi\, d\dot{\boldsymbol{R}}_1\, d\dot{\boldsymbol{R}}_2, \tag{11.2.5}$$

and this average is a function of $\boldsymbol{R}_1, \boldsymbol{R}_2$ and t; its phase space average is given by

$$\langle B\rangle = \frac{1}{nV}\iint [\![B]\!](\boldsymbol{R}_1, \boldsymbol{R}_2, t)\Psi\, d\boldsymbol{R}_1\, d\boldsymbol{R}_2, \quad V = \text{volume of the solution}, \tag{11.2.6}$$

and this average is a function of t only.

(a) **External force $\boldsymbol{F}_\nu^e$.** Examples are gravitational and electrical forces.

(b) **Intramolecular force $\boldsymbol{F}_\nu^\phi$.** This is the force on one bead resulting from the spring in the dumbbell, given by

$$\boldsymbol{F}_\nu^\phi = -\frac{\partial}{\partial \boldsymbol{R}_\nu}\phi, \tag{11.2.7}$$

where ϕ is the spring potential energy.

(c) **Hydrodynamic drag force $\boldsymbol{F}_\nu^h$.** This is the drag force experienced by a bead as it moves through the solution. Similar to Stokes' law, the drag force is proportional to the difference between the bead velocity (appropriately averaged with respect to the velocity distribution) and the mass-average velocity of the solution. More precisely,

$$\boldsymbol{F}_\nu^h = -\underline{\underline{\xi}}\cdot\left([\![\dot{\boldsymbol{R}}_\nu]\!] - (\boldsymbol{v}_\nu + \boldsymbol{v}'_\nu)\right), \tag{11.2.8}$$

where $\boldsymbol{v}_\nu = \boldsymbol{v}_0 + \underline{\underline{\kappa}}\cdot\boldsymbol{R}_\nu$ is the imposed homogeneous flow field at bead ν, $\boldsymbol{v}'_\nu$ the perturbation of the flow field at bead ν due to the motion of the other bead and $\underline{\underline{\xi}}$ is a symmetric second-order tensor, called the drag tensor. Note that evaluating $[\![\dot{\boldsymbol{R}}_\nu]\!]$ using the Maxwell velocity distribution (11.2.4) would give the fluid velocity $\boldsymbol{v}$.

(d) **Brownian force $\boldsymbol{F}_\nu^b$.** This is due to the thermal fluctuations in the liquid. Because the true Brownian motion force is rapidly and irregularly fluctuating, we use a statistically averaged force instead. More precisely, it takes the form

$$\boldsymbol{F}_\nu^b = -\frac{1}{\Psi}\frac{\partial}{\partial \boldsymbol{R}_\nu}\left([\![m(\boldsymbol{R}_\nu - \boldsymbol{v}\mathbb{R})(\boldsymbol{R}_\nu - \boldsymbol{v}\mathbb{R})]\!]\Psi\right).$$

Note that this is the divergence of a momentum flux with respect to the fluid velocity $\boldsymbol{v}$ at the center of mass of the dumbbell. Almost all kinetic theories assume equilibration in momentum space (11.2.4), in this case the Brownian force assumes a much simpler form:

$$\boldsymbol{F}^b_\nu = -kT\frac{\partial}{\partial \boldsymbol{R}_\nu}\ln\Psi, \tag{11.2.9}$$

and we will use the above expression for $\boldsymbol{F}^b_\nu$ unless stated otherwise.

In addition to the previous four assumptions about the configuration-space and velocity-space distribution functions, we also impose four assumptions about these forces:

1. Inertia of the beads is negligible, *i.e.* the motion of each bead is dominated by drag and viscous effect of the solution. FIt follows from Newton's second law that

$$\boldsymbol{F}^h_\nu + \boldsymbol{F}^b_\nu + \boldsymbol{F}^\phi_\nu + \boldsymbol{F}^e_\nu = \boldsymbol{0}. \tag{11.2.10}$$

2. The external forces $\boldsymbol{F}^e_\nu$ are independent of $\boldsymbol{R}_c$.

3. The hydrodynamic interaction, *i.e.* hydrodynamic forces due to the motion of the other bead, is negligible. This means that $\boldsymbol{v}'_\nu \approx \boldsymbol{0}$.

4. The friction tensor $\underline{\underline{\xi}}$ is assumed to be isotropic, *i.e.* $\underline{\underline{\xi}} = \xi\underline{\underline{I}}$, ξ being the drag coefficient. This together with the previous assumption simplifies the hydrodynamic force to

$$\boldsymbol{F}^h_\nu = -\xi\Big([\![\dot{\boldsymbol{R}}_\nu]\!] - \boldsymbol{v}_\nu\Big) \tag{11.2.11}$$

11.2.2 Equation of motion for the beads

We may now derive the PDE for the configurational-space distribution function. Substituting(11.2.1), (11.2.11), (11.2.9) and (11.2.3) into (11.2.10) we obtain the equation of motion for the two beads:

$$-\xi\Big([\![\dot{\boldsymbol{R}}_\nu]\!] - \boldsymbol{v}_0 - \underline{\underline{\kappa}}\cdot\boldsymbol{R}_\nu\Big) - kT\frac{\partial}{\partial \boldsymbol{R}_\nu}\ln\psi + \boldsymbol{F}^\phi_\nu + \boldsymbol{F}^e_\nu = \boldsymbol{0}, \quad \nu = 1, 2. \tag{11.2.12}$$

For the dumbbell models the intramolecular forces on the two beads are equal and opposite, so we define a connector force $\boldsymbol{F}^c$ by $\boldsymbol{F}^c = \boldsymbol{F}^\phi_1 = -\boldsymbol{F}^\phi_2$. Adding (11.2.12) and recalling the center of mass $\boldsymbol{R}_c = (\boldsymbol{R}_1 + \boldsymbol{R}_2)/2$, we obtain the equation of motion for $\boldsymbol{R}_c$:

$$\begin{aligned}
-\xi\Big([\![\dot{\boldsymbol{R}}_1]\!] + [\![\dot{\boldsymbol{R}}_2]\!] - 2\boldsymbol{v}_0 - \underline{\underline{\kappa}}\cdot(\boldsymbol{R}_1+\boldsymbol{R}_2)\Big) - kT\left(\frac{\partial}{\partial \boldsymbol{R}_1}\ln\psi + \frac{\partial}{\partial \boldsymbol{R}_2}\ln\psi\mathbb{R}\right) + \boldsymbol{F}^e_1 + \boldsymbol{F}^e_2 = \boldsymbol{0}\\
-2\xi\Big([\![\dot{\boldsymbol{R}}_c]\!] - 2\boldsymbol{v}_0 - 2\underline{\underline{\kappa}}\cdot\boldsymbol{R}_c\Big) - kT\left(\frac{\partial}{\partial \boldsymbol{R}_1}\psi + \frac{\partial}{\partial \boldsymbol{R}_2}\psi\mathbb{R}\right) + \boldsymbol{F}^e_1 + \boldsymbol{F}^e_2 = \boldsymbol{0}.
\end{aligned}$$

Since chain rule gives

$$\frac{\partial}{\partial \boldsymbol{R}_\nu}\ln\psi = \left(\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{R}_\nu}\mathbb{R}\right)\cdot\left(\frac{\partial}{\partial \boldsymbol{Q}}\ln\psi\mathbb{R}\right) = \begin{cases} -\dfrac{\partial}{\partial \boldsymbol{Q}}\ln\psi & \text{if } \nu = 1,\\ \dfrac{\partial}{\partial \boldsymbol{Q}}\ln\psi & \text{if } \nu = 2,\end{cases}$$

the equation of motion for $\boldsymbol{R}_c$ reduces to

$$[\![\dot{\boldsymbol{R}}_c]\!] = \boldsymbol{v}_0 + \underline{\underline{\kappa}} \cdot \boldsymbol{R}_c + \frac{1}{2\xi}\Big(\boldsymbol{F}_1^e + \boldsymbol{F}_2^e\Big). \tag{11.2.13}$$

On the other hand, subtracting (11.2.12) yields

$$-\xi\Big([\![\dot{\boldsymbol{R}}_2]\!] - [\![\dot{\boldsymbol{R}}_1]\!] - \underline{\underline{\kappa}} \cdot (\boldsymbol{R}_2 - \boldsymbol{R}_1)\Big) - kT\left(\frac{\partial}{\partial \boldsymbol{R}_2}\ln\psi - \frac{\partial}{\partial \boldsymbol{R}_1}\ln\psi\mathbb{R}\right) + \boldsymbol{F}_2^\phi - \boldsymbol{F}_1^\phi + \boldsymbol{F}_2^e - \boldsymbol{F}_1^e = \boldsymbol{0}$$
$$-\xi\Big([\![\dot{\boldsymbol{Q}}]\!] - \underline{\underline{\kappa}} \cdot \boldsymbol{Q}\Big) - 2kT\left(\frac{\partial}{\partial \boldsymbol{Q}}\ln\psi\mathbb{R}\right) - 2\boldsymbol{F}^c + \boldsymbol{F}_2^e - \boldsymbol{F}_1^e = \boldsymbol{0}.$$

Rearranging gives the equation of motion for the connector vector $\boldsymbol{Q}$:

$$[\![\dot{\boldsymbol{Q}}]\!] = \underline{\underline{\kappa}} \cdot \boldsymbol{Q} - \frac{2kT}{\xi}\left(\frac{\partial}{\partial \boldsymbol{Q}}\ln\psi\mathbb{R}\right) - \frac{2}{\xi}\boldsymbol{F}^c + \frac{1}{\xi}\Big(\boldsymbol{F}_2^e - \boldsymbol{F}_1^e\Big). \tag{11.2.14}$$

11.2.3 Equation of continuity for $\psi(\boldsymbol{Q}, t)$

Viewing $\boldsymbol{R}_1, \boldsymbol{R}_2$ as a single point in a six-dimensional configuration space, one can show that

$$\begin{aligned}
\partial_t \Psi &= -\left(\frac{\partial}{\partial \boldsymbol{R}_1} \cdot \Big([\![\dot{\boldsymbol{R}}_1]\!]\Psi\mathbb{R}\Big)\mathbb{R}\right) - \left(\frac{\partial}{\partial \boldsymbol{R}_2} \cdot \Big([\![\dot{\boldsymbol{R}}_2]\!]\Psi\mathbb{R}\Big)\mathbb{R}\right) \\
&= -\left(\frac{\partial}{\partial \boldsymbol{R}_c} \cdot \Big([\![\dot{\boldsymbol{R}}_c]\!]\Psi\mathbb{R}\Big)\mathbb{R}\right) - \left(\frac{\partial}{\partial \boldsymbol{Q}} \cdot \Big([\![\dot{\boldsymbol{Q}}]\!]\Psi\mathbb{R}\Big)\mathbb{R}\right) \\
&= -\left(\frac{\partial}{\partial \boldsymbol{Q}} \cdot [\![\dot{\boldsymbol{Q}}]\!]\Psi\mathbb{R}\right).
\end{aligned}$$

The first term vanishes since from (11.2.3) and (11.2.13) we have

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{R}_c} \cdot \Big([\![\dot{\boldsymbol{R}}_c]\!]\Psi\mathbb{R}\Big) &= [\![\dot{\boldsymbol{R}}_c]\!] \cdot \left(2\frac{\partial\Psi}{\partial \boldsymbol{R}_c}\mathbb{R}\right) + \Psi\frac{\partial}{\partial \boldsymbol{R}_c} \cdot [\![\dot{\boldsymbol{R}}_c]\!] \\
&= [\![\dot{\boldsymbol{R}}_c]\!] \cdot \boldsymbol{0} + \Psi\mathrm{tr}\left(\underline{\underline{\kappa}}\mathbb{R}\right) \\
&= 0,
\end{aligned}$$

where we use the assumption that ψ is independent of $\boldsymbol{R}_c$ and $\underline{\underline{\kappa}}$ is traceless. Finally, substituting (11.2.14) for $[\![\dot{\boldsymbol{Q}}]\!]$ we obtain the **diffusion equation** for $\psi(\boldsymbol{Q}, t)$:

$$\partial_t\psi = -\frac{\partial}{\partial \boldsymbol{Q}} \cdot \left(\Big[\underline{\underline{\kappa}} \cdot \boldsymbol{Q}\Big]\psi - \frac{2kT}{\xi}\left(\frac{\partial\psi}{\partial \boldsymbol{Q}}\mathbb{R}\right) - \frac{2}{\xi}\boldsymbol{F}^c\psi + \frac{1}{\xi}\Big[\boldsymbol{F}_2^e - \boldsymbol{F}_1^e\Big]\psi\mathbb{R}\right). \tag{11.2.15}$$

Remark 11.2.1. We can multiply (11.2.15) by any function $B(\boldsymbol{Q})$ and integrate over all the configuration space as in (11.2.6). This gives the evolution equation for the phase space average $\langle B \rangle$ of B:

$$\begin{aligned}
\frac{d}{dt}\langle B \rangle = \underline{\underline{\kappa}} \colon \left\langle \boldsymbol{Q}\frac{\partial B}{\partial \boldsymbol{Q}}\mathbb{R}\right\rangle + \frac{2kT}{\xi}\left\langle \frac{\partial}{\partial \boldsymbol{Q}} \cdot \frac{\partial B}{\partial \boldsymbol{Q}}\mathbb{R}\right\rangle \\
- \frac{2}{\xi}\left\langle \boldsymbol{F}^c \cdot \frac{\partial B}{\partial \boldsymbol{Q}}\mathbb{R}\right\rangle + \frac{1}{\xi}\left\langle \Big[\boldsymbol{F}_2^e - \boldsymbol{F}_1^e\Big] \cdot \frac{\partial B}{\partial \boldsymbol{Q}}\mathbb{R}\right\rangle.
\end{aligned}$$

11.3 Expressions for the Stress Tensor

The total stress tensor $\underline{\underline{\sigma}}$ in a polymer solution is presumed to be the sum of a contribution from the solvent, $\underline{\underline{\sigma}}_s$ and the presence of polymer molecules, $\underline{\underline{\sigma}}_p$:

$$\begin{aligned}\underline{\underline{\sigma}} &= \underline{\underline{\sigma}}_s + \underline{\underline{\sigma}}_p \\ &= \Big(-p_s\underline{\underline{I}} + \underline{\underline{\tau}}_s\Big) + \Big(p_p\underline{\underline{I}} + \underline{\underline{\tau}}_p\Big) \\ &= -p\underline{\underline{I}} + \underline{\underline{\tau}},\end{aligned}$$

where $p = p_s - p_p$, $\underline{\underline{\tau}} = \underline{\underline{\tau}}_s + \underline{\underline{\tau}}_p$ and $\underline{\underline{\tau}}_s = 2\eta_s\underline{\underline{E}}$, η_s being the solvent viscosity. Kramers showed that there are 3 main contributions to $\underline{\underline{\sigma}}_s$:

$$\underline{\underline{\sigma}}_p = \underline{\underline{\pi}}_p^c + \underline{\underline{\pi}}_p^e + \underline{\underline{\pi}}_p^b,$$

where

$$\begin{aligned}\underline{\underline{\pi}}_p^c &= \text{tension or compression force transmitted along} \\ &\quad\ \text{the connector due to straddling of dumbbells} \\ \underline{\underline{\pi}}_p^e &= \text{effects due to external forces acting on the dumbbells} \\ &\quad\ \text{straddling the plane in the solution} \\ \underline{\underline{\pi}}_p^b &= \text{stress due to the bead motion across the plane.}\end{aligned}$$

11.3.1 Contribution from the intramolecular potential

Consider an arbitrary plane of area S in the solution moving with local velocity $\boldsymbol{v}$. The orientation of the plane is given by a unit normal vector $\boldsymbol{n}$. How many dumbbells with connector vector $\boldsymbol{Q}$ will be straddling the plane, with bead ① on the ⊖ side and bead ② on the ⊕ side? Let

$$\begin{aligned} n &= \text{the number of dumbbells per unit volume } V \\ \boldsymbol{n}\cdot\boldsymbol{Q}S &= \text{the volume in which bead ① must be} \\ \psi(\boldsymbol{Q},t)\,d\boldsymbol{Q} &= \text{probability that the dumbbell is in the configuration range } d\boldsymbol{Q} \text{ about } \boldsymbol{Q}.\end{aligned}$$

There will be a contribution to the force of the negative material on the positive material in the amount of $-\boldsymbol{F}_1^\phi$, so the stress contribution of dumbbells of all orientation with bead ① on the negative side and bead ② on the positive side is

$$\frac{1}{S}\int_{\boldsymbol{Q}\text{ such that }\boldsymbol{n}\cdot\boldsymbol{Q}>0} n\,(\boldsymbol{n}\cdot\boldsymbol{Q}\mathbb{R})\,S(-\boldsymbol{F}_1^\phi)\psi(\boldsymbol{Q},t)\,d\boldsymbol{Q}.$$

Similarly, the stress contribution of the dumbbells of all orientations with bead ② on the negative side and bead ① on the positive side is

$$\frac{1}{S}\int_{\boldsymbol{Q}\text{ such that }\boldsymbol{n}\cdot\boldsymbol{Q}<0} n\,(-\boldsymbol{n}\cdot\boldsymbol{Q}\mathbb{R})\,S(-\boldsymbol{F}_2^\phi)\psi(\boldsymbol{Q},t)\,d\boldsymbol{Q}.$$

Hence the total contribution to the stress of the negative material on the positive material is

$$\begin{aligned}
&-\int_{\boldsymbol{n}\cdot\boldsymbol{Q}>0} n\,(\boldsymbol{n}\cdot\boldsymbol{Q}\mathbb{R})\,\boldsymbol{F}^c\psi(\boldsymbol{Q},t)\,dQ - \int_{\boldsymbol{n}\cdot\boldsymbol{Q}<0} n\,(\boldsymbol{n}\cdot\boldsymbol{Q}\mathbb{R})\,\boldsymbol{F}^c\psi(\boldsymbol{Q},t)\,dQ\\
&= -\int_{\boldsymbol{Q}} n\,(\boldsymbol{n}\cdot\boldsymbol{Q}\mathbb{R})\,\boldsymbol{F}^c\psi(\boldsymbol{Q},t)\,dQ\\
&= -\left[\boldsymbol{n}\cdot n\int \boldsymbol{Q}\boldsymbol{F}^c\psi(\boldsymbol{Q},t)\,dQ\mathbb{R}\right]\\
&= -\boldsymbol{n}\cdot\Big(n\langle\boldsymbol{Q}\boldsymbol{F}^c\rangle\Big),
\end{aligned}$$

but this equals to $\boldsymbol{n}\cdot\underline{\underline{\pi}}_p^c$, the traction from the fluid acting on the surface with unit normal vector $\boldsymbol{n}$. Hence,

$$\underline{\underline{\pi}}_p^c = -n\langle\boldsymbol{Q}\boldsymbol{F}^c\rangle.$$

Since the force $\boldsymbol{F}^c$ is directed along the connector vector $\boldsymbol{Q}$, we can rewrite it as $\boldsymbol{F}^c = |\boldsymbol{F}^c|\boldsymbol{Q}/|\boldsymbol{Q}|$, where $|\boldsymbol{F}^c|$ is the magnitude of $\boldsymbol{F}^c$. Consequently, $\underline{\underline{\pi}}_p^c$ is a symmetric tensor. This also suggests that we need to compute $\langle\boldsymbol{Q}\boldsymbol{Q}\rangle$, the phase space average of the second-order tensor $\boldsymbol{Q}\boldsymbol{Q}$. One can use the diffusion equation (11.2.15) for $\psi(\boldsymbol{Q},t)$ to obtain the evolution equation for $\langle\boldsymbol{Q}\boldsymbol{Q}\rangle$:

$$\begin{aligned}
\frac{d}{dt}\langle\boldsymbol{Q}\boldsymbol{Q}\rangle - \underline{\underline{\kappa}}\cdot\langle\boldsymbol{Q}\boldsymbol{Q}\rangle - \langle\boldsymbol{Q}\boldsymbol{Q}\rangle\cdot\underline{\underline{\kappa}}^T &= \frac{4kT}{\xi}\underline{\underline{I}} - \frac{4}{\xi}\langle\boldsymbol{Q}\boldsymbol{F}^c\rangle\\
&+ \frac{1}{\xi}\langle(\boldsymbol{F}_2^e - \boldsymbol{F}_1^e\mathbb{R})\,\boldsymbol{Q} + \boldsymbol{Q}\,(\boldsymbol{F}_2^e - \boldsymbol{F}_1^e\mathbb{R})\,\boldsymbol{Q}\rangle. \qquad (11.3.1)
\end{aligned}$$

The second-order tensor

$$\langle\boldsymbol{Q}\boldsymbol{Q}\rangle_1 = \frac{d}{dt}\langle\boldsymbol{Q}\boldsymbol{Q}\rangle - \underline{\underline{\kappa}}\cdot\langle\boldsymbol{Q}\boldsymbol{Q}\rangle - \langle\boldsymbol{Q}\boldsymbol{Q}\rangle\cdot\underline{\underline{\kappa}}^T \qquad (11.3.2)$$

is known as the upper convective derivative.

11.3.2 Contribution from the external forces

Similar to the derivation of $\underline{\underline{\pi}}_p^c$, the stress contribution of the negative material on the positive material is

$$\int_{\boldsymbol{n}\cdot\boldsymbol{Q}>0} n\,(\boldsymbol{n}\cdot\boldsymbol{Q}\mathbb{R})\,(-\boldsymbol{F}_1^e)\psi(\boldsymbol{Q},t)\,dQ + \int_{\boldsymbol{n}\cdot\boldsymbol{Q}<0} n\,(-\boldsymbol{n}\cdot\boldsymbol{Q}\mathbb{R})\,(-\boldsymbol{F}_2^e)\psi(\boldsymbol{Q},t)\,dQ. \qquad (11.3.3)$$

On the other hand, the stress contribution of the positive material on the negative material is

$$\int_{\boldsymbol{n}\cdot\boldsymbol{Q}>0} n\,(\boldsymbol{n}\cdot\boldsymbol{Q}\mathbb{R})\,(-\boldsymbol{F}_2^e)\psi(\boldsymbol{Q},t)\,dQ + \int_{\boldsymbol{n}\cdot\boldsymbol{Q}<0} n\,(-\boldsymbol{n}\cdot\boldsymbol{Q}\mathbb{R})\,(-\boldsymbol{F}_1^e)\psi(\boldsymbol{Q},t)\,dQ, \qquad (11.3.4)$$

and this must be the negative of (11.3.3). Hence, substracting (11.3.4) from (11.3.3) and halving the resulting expression we obtain

$$\boldsymbol{n}\cdot\left(\frac{1}{2}n\langle\boldsymbol{Q}\,(\boldsymbol{F}_2^e - \boldsymbol{F}_1^e\mathbb{R})\rangle\mathbb{R}\right) = \boldsymbol{n}\cdot\underline{\underline{\pi}}_p^e,$$

which in turn gives

$$\underline{\underline{\pi}}_p^e = \frac{1}{2}n\langle\boldsymbol{Q}\,(\boldsymbol{F}_2^e - \boldsymbol{F}_1^e\mathbb{R})\rangle.$$

Note that this contribution is not necessarily symmetric.

11.3.3 Contribution from the bead motion

The motion of the beads across an arbitrary plane will contribute to the stress tensor because of the momentum transported by the beads. How many ① beads with velocity $\dot{\boldsymbol{R}}_1$ will cross an arbitrary surface in time Δt? This is given by

$$n \underbrace{\left[\left(\dot{\boldsymbol{R}}_1 - \boldsymbol{v}\mathbb{R} \right) \cdot S\boldsymbol{n} \right] \Delta t}_{\text{volume}} .$$

The amount of momentum transported is then

$$n\left[\left(\dot{\boldsymbol{R}}_1 - \boldsymbol{v}\mathbb{R} \right) \cdot S\boldsymbol{n} \right] m \left(\dot{\boldsymbol{R}}_1 - \boldsymbol{v}\mathbb{R} \right) \Delta t,$$

and the average value of the momentum flux (momentum per unit time per unit area) resulting from beads ① is

$$n\boldsymbol{n} \cdot \int [\![m \left(\dot{\boldsymbol{R}}_1 - \boldsymbol{v}\mathbb{R} \right) \left(\dot{\boldsymbol{R}}_1 - \boldsymbol{v}\mathbb{R} \right)]\!] \psi(\boldsymbol{Q}, t)\, d\boldsymbol{Q}. \tag{11.3.5}$$

A similar expression holds for the average value of the momentum flux resulting from beads ②. Thus the average value of the momentum flux resulting from both beads is obtained by adding these two contributions, and it must equal to $\boldsymbol{n} \cdot \underline{\underline{\pi}}_p^b$. We obtain

$$\underline{\underline{\pi}}_p^b = n \int \left[\!\!\left[\sum_{\nu=1}^{2} m \left(\dot{\boldsymbol{R}}_\nu - \boldsymbol{v}\mathbb{R} \right) \left(\dot{\boldsymbol{R}}_\nu - \boldsymbol{v}\mathbb{R} \right) \mathbb{R} \right]\!\!\right] \psi(\boldsymbol{Q}, t)\, d\boldsymbol{Q}.$$

For a Maxwellian velocity distribution,

$$\underline{\underline{\pi}}_p^b = 2nkT\underline{\underline{I}},$$

and this contribution has no rheological impact since it is isotropic.

11.3.4 Summary

Finally, the total stress tensor of a dilute solution of dumbbells with the Maxwellian velocity distribution is

$$\begin{aligned}
\underline{\underline{\sigma}} &= \underline{\underline{\sigma}}_s + \underline{\underline{\sigma}}_p \\
&= \underline{\underline{\sigma}}_s - n\langle \boldsymbol{Q}\boldsymbol{F}^c \rangle + \frac{1}{2} n \langle \boldsymbol{Q} \left(\boldsymbol{F}_2^e - \boldsymbol{F}_1^e \mathbb{R} \right) \rangle + 2nkT\underline{\underline{I}} \\
&= \underline{\underline{\sigma}}_s + n \sum_{\nu=1}^{2} \langle \boldsymbol{R}_\nu \left(\boldsymbol{F}_\nu^\phi + \boldsymbol{F}_\nu^e \mathbb{R} \right) \rangle + 2nkT\underline{\underline{I}},
\end{aligned}$$

where $\boldsymbol{R}_\nu = \boldsymbol{R}_\nu - \boldsymbol{R}_c$ is the location of the beads relative to the center of mass and $\boldsymbol{F}_\nu^\phi = -\dfrac{\partial \phi}{\partial \boldsymbol{R}_\nu}$.

In a system at equilibrium, that is $\underline{\underline{\kappa}} = \underline{\underline{\boldsymbol{0}}}$ and $\boldsymbol{F}_\nu^e = \boldsymbol{0}$, from (11.3.1) we obtain

$$\underline{\underline{0}} = \frac{4kT}{\xi} \underline{\underline{I}} - \frac{4}{\xi} \langle \boldsymbol{Q}\boldsymbol{F}^c \rangle$$

$$\langle \boldsymbol{Q}\boldsymbol{F}^c \rangle = kT\underline{\underline{I}},$$

and so

$$\begin{aligned}
-p\underline{\underline{I}} &= -p_s\underline{\underline{I}} - n\langle \boldsymbol{Q}\boldsymbol{F}^c \rangle + 2nkT\underline{\underline{I}} \\
&= -p_s\underline{\underline{I}} - nkT\underline{\underline{I}} + 2nkT\underline{\underline{I}} \\
&= -p_s\underline{\underline{I}} + nkT\underline{\underline{I}},
\end{aligned}$$

i.e. $p_p = nkT$. Consequently,

$$\begin{aligned}
\underline{\underline{\tau}}_p = \underline{\underline{\sigma}}_p - p_p\underline{\underline{I}} &= -n\langle \boldsymbol{Q}\boldsymbol{F}^c \rangle + \frac{1}{2}n\langle \boldsymbol{Q}\left(\boldsymbol{F}_2^e - \boldsymbol{F}_1^e\mathbb{R}\right)\rangle + 2nkT\underline{\underline{I}} - nkT\underline{\underline{I}} \\
&= -n\langle \boldsymbol{Q}\boldsymbol{F}^c \rangle + \frac{1}{2}n\langle \boldsymbol{Q}\left(\boldsymbol{F}_2^e - \boldsymbol{F}_1^e\mathbb{R}\right)\rangle + nkT\underline{\underline{I}}.
\end{aligned}$$

Recall the stress tensor $\underline{\underline{\tau}} = \underline{\underline{\tau}}_s + \underline{\underline{\tau}}_p$. For a Newtonian solvent, $\underline{\underline{\tau}}_s = 2\eta_s\underline{\underline{E}}$ and we obtain the **Kramers stress tensor**:

$$\underline{\underline{\tau}} = 2\eta_s\underline{\underline{E}} - n\langle \boldsymbol{Q}\boldsymbol{F}^c \rangle + \frac{1}{2}n\langle \boldsymbol{Q}\left(\boldsymbol{F}_2^e - \boldsymbol{F}_1^e\mathbb{R}\right)\rangle + nkT\underline{\underline{I}} \tag{11.3.6}$$

The **modified Kramers stress tensor** takes the alternate form:

$$\underline{\underline{\tau}} = 2\eta_s\underline{\underline{E}} + n\sum_{\nu=1}^{2}\langle \boldsymbol{R}_\nu\left(\boldsymbol{F}_\nu^\phi + \boldsymbol{F}_\nu^e\mathbb{R}\right)\rangle + nkT\underline{\underline{I}}. \tag{11.3.7}$$

Because the sum of forces equals 0, from (11.2.10) we have

$$\boldsymbol{F}_\nu^\phi + \boldsymbol{F}_\nu^e = -\boldsymbol{F}_\nu^h - \boldsymbol{F}_\nu^b.$$

Substituting the Brownian force (11.2.9) and integrating by parts then gives

$$\begin{aligned}
\underline{\underline{\tau}} &= 2\eta_s\underline{\underline{E}} + n\sum_{\nu=1}^{2}\langle \boldsymbol{R}_\nu\left(\boldsymbol{F}_\nu^\phi + \boldsymbol{F}_\nu^e\mathbb{R}\right)\rangle + nkT\underline{\underline{I}} \\
&= 2\eta_s\underline{\underline{E}} + n\sum_{\nu=1}^{2}\langle \boldsymbol{R}_\nu\left(-\boldsymbol{F}_\nu^h - \boldsymbol{F}_\nu^b\mathbb{R}\right)\rangle + nkT\underline{\underline{I}} \\
&= 2\eta_s\underline{\underline{E}} - n\sum_{\nu=1}^{2}\langle \boldsymbol{R}_\nu\boldsymbol{F}_\nu^b\rangle,
\end{aligned}$$

called the **Kramers-Kirkwood stress tensor**.

11.4 Hookean Dumbbells

We investigate the specific case of a Hookean spring connector for which $\boldsymbol{F}^c = H\boldsymbol{Q}$, H the spring constant. This is known as the **Oldroyd-B model**. For this model the polymer contribution to the stress tensor $\underline{\underline{\tau}}_p$ takes the form:

$$\underline{\underline{\tau}}_p = -nH\langle \boldsymbol{Q}\boldsymbol{Q}\rangle + nkT\underline{\underline{I}} \tag{Kramers}$$

$$\underline{\underline{\tau}}_p = \frac{n\xi}{4}\langle \overset{\triangledown}{\boldsymbol{QQ}} \rangle \qquad \text{(Giesekus)}$$

We can eliminate $\langle \boldsymbol{QQ} \rangle$ from these two equations. Let $\lambda_H = \xi/4H$ be the time constant for the Hookean dumbbells.

$$\begin{aligned} \underline{\underline{\tau}}_p - nkT\underline{\underline{I}} &= -nH\langle \boldsymbol{QQ} \rangle \\ \overset{\triangledown}{\underline{\underline{\tau}}}_p - nkT\overset{\triangledown}{\underline{\underline{I}}} &= -nH\langle \overset{\triangledown}{\boldsymbol{QQ}} \rangle = -\frac{4H}{\xi}\underline{\underline{\tau}}_p. \end{aligned}$$

Since

$$\begin{aligned} \overset{\triangledown}{\underline{\underline{I}}} &= \frac{d}{dt}\underline{\underline{I}} - \underline{\underline{\kappa}} \cdot \underline{\underline{I}} - \underline{\underline{I}} \cdot \underline{\underline{\kappa}}^T \\ &= -\underline{\underline{\kappa}} - \underline{\underline{\kappa}}^T \\ &= -\nabla \boldsymbol{v} - \nabla \boldsymbol{v}^T \\ &= -2\underline{\underline{E}}, \end{aligned}$$

we obtain

$$\begin{aligned} \overset{\triangledown}{\underline{\underline{\tau}}}_p + \frac{4H}{\xi}\underline{\underline{\tau}}_p &= -2nkT\underline{\underline{E}} \\ \lambda\overset{\triangledown}{\underline{\underline{\tau}}}_p + \underline{\underline{\tau}}_p &= -2nkT\lambda\underline{\underline{E}}. \end{aligned}$$

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